An Overview of Matrices

1 Matrix Definition

A **matrix** A is a rectangular array of numbers (or functions), which are called the **entries** (or **elements**) of the matrix. If A has m rows and n columns, it is said to have **size** $m \times n$ (pronounced "m by n").

A double set of subscripts is used to refer to the entries. The entry in row i and column j of a matrix A is denoted a_{ij}. In other words

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

2 Square Matrix

An $m \times n$ matrix is **square** if $m = n$. The entries $a_{11}, a_{22}, \ldots, a_{nn}$ of a square matrix are called its **diagonal entries**. For example, matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

is a square 3×3 matrix with diagonal entries 1, 5 and 9. The matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

is a 2×3 matrix and is not square.

3 Equality

Two matrices A and B are said to be **equal** if they have the same size and if corresponding entries are equal; in other words, $a_{ij} = b_{ij}$ for all i and j.

4 Zero and Identity Matrices

A matrix whose entries are all zero is called the **zero matrix** and is denoted O. For example,

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is the 2×3 zero matrix. It may be denoted O_{23} to emphasize its size.

A square matrix whose diagonal entries are all 1 and whose other entries are all 0 is called the **identity matrix** and is denoted I. For example,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the 3×3 identity matrix. It may be denoted I_3 to emphasize its size.

5 Transpose

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T formed by interchanging the rows and columns of A. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},$$

then

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

If A is a square matrix, then it is said to be **symmetric** if $A^T = A$.

6 Matrix Operations

Arithmetic operations can be performed on matrices, including scalar multiplication, addition, subtraction and multiplication.

Scalar multiplication of a matrix by a constant (or function) k is performed by multiplying each entry by k. For example, if

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 6 & 2 & -7 \end{bmatrix},$$

and $k = -3$, then

$$-3A = \begin{bmatrix} -3 & 6 & 0 \\ -18 & -6 & 21 \end{bmatrix}.$$

Addition and **subtraction** of matrices are done by adding or subtracting their respective entries. For example,

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} + \begin{bmatrix}1 & 0 & -3\\ 2 & 7 & -1\end{bmatrix} = \begin{bmatrix}2 & 2 & 0\\ 6 & 12 & 5\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}7 & 2\\ 0 & -3\end{bmatrix} - \begin{bmatrix}3 & -5\\ 1 & 1\end{bmatrix} = \begin{bmatrix}4 & 7\\ -1 & -4\end{bmatrix}.$$

The sum or difference of two matrices is only defined when the matrices have the same size. Note that $A + O = A$ and $O + A = A$.

Multiplication of two matrices A and B is only defined when the number of columns of A is equal to the number or rows of B, such as when A is an $m \times n$ matrix and B is an $n \times p$ matrix. The resulting product $C = AB$ will be an $m \times p$ matrix. Each entry c_{ij} of C is found by summing the products of the entries of row i in A with the corresponding entries of column j in B. In other words,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

For example,

$$AB = \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix} \begin{bmatrix}1 & -2\\ 5 & -3\end{bmatrix} = \begin{bmatrix}1 \cdot 1 + 2 \cdot 5 & 1 \cdot (-2) + 2 \cdot (-3)\\ 3 \cdot 1 + 4 \cdot 5 & 3 \cdot (-2) + 4 \cdot (-3)\\ 5 \cdot 1 + 6 \cdot 5 & 5 \cdot (-2) + 6 \cdot (-3)\end{bmatrix} = \begin{bmatrix}11 & -8\\ 23 & -18\\ 35 & -28\end{bmatrix}.$$

In this example, the product BA is undefined since the sizes are incompatible. Even when both products AB and BA are defined, in the case where A and B are both $n \times n$ square matrices, in general $AB \neq BA$; in other words matrix multiplication is not commutative.

Matrix multiplication is, nevertheless, associative, so that $A(BC) = (AB)C$ and it distributes over matrix addition, so that $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$.

If A is $m \times n$, then $AI_n = A$ and $I_m A = A$, where I_n and I_m are the $n \times n$ and $m \times m$ identity matrices, respectively.

7 Systems of Equations

A n-dimensional vector $\mathbf{x}$ can be represented by an $n \times 1$ matrix having only one column,

$$\mathbf{x} = \begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_n\end{bmatrix}.$$

A system of equations

$$\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \ a_{13}x_3 + \cdots + \ a_{1n}x_n &= b_1 \\
a_{21}x_1 + \ a_{22}x_2 + \ a_{23}x_3 + \cdots + \ a_{2n}x_n &= b_2 \\
a_{31}x_1 + \ a_{32}x_2 + \ a_{33}x_3 + \cdots + \ a_{3n}x_n &= b_3 \\
\vdots \qquad\quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

can be written in the form $A\mathbf{x} = \mathbf{b}$, given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}.$$

8 Derivatives and Integrals

If the entries of a matrix A or a vector $\mathbf{x}$ are functions of t, then the derivative and integral of $A(t)$ or $\mathbf{x}(t)$ are found by differentiating and integrating, respectively, each entry. For example, if

$$A(t) = \begin{bmatrix} t & 3t^2 \\ t^3 & 5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}(t) = \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix},$$

then

$$A'(t) = \frac{dA}{dt} = \begin{bmatrix} 1 & 6t \\ 3t^2 & 0 \end{bmatrix}, \qquad \mathbf{x}'(t) = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2\cos 2t \\ -2\sin 2t \end{bmatrix},$$

$$\int A(t)\,dt = \begin{bmatrix} \frac{1}{2}t^2 & t^3 \\ \frac{1}{4}t^4 & 5t \end{bmatrix} + C, \qquad \int \mathbf{x}(t)\,dt = \begin{bmatrix} -\frac{1}{2}\cos 2t \\ \frac{1}{2}\sin 2t \end{bmatrix} + \mathbf{c},$$

where C is an arbitrary 2×2 constant matrix and $\mathbf{c}$ is an arbitrary 2×1 constant vector.

9 Determinants

Associated with each square matrix is a real number called its **determinant**, denoted $\det(A)$ or $|A|$. For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

its determinant is given by the formula

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Determinants of larger matrices can be calculated recursively (though not necessarily efficiently) using the **method of minors**. We begin by associating with an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix},$$

a similarly sized matrix of alternating signs of the form

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

beginning with $+$ in the upper left. We can then calculate $\det(A)$ by expanding along any single row or column of the matrix. This is done by taking each entry a_{ij} in the chosen row or column, multiplying it by $(-1)^{i+j}$, which is equivalent to applying the associated sign from the matrix of alternating signs, and then multiplying it further by the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the row i and column j of matrix A to which the entry a_{ij} belongs (such a matrix is called a **submatrix** and its determinant is known as a **minor**). These products are then summed together to form $\det(A)$.

For example, suppose

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & 1 & 7 \\ -3 & 2 & 5 \end{bmatrix}.$$

Expanding across the first row gives us

$$\det(A) = 1 \begin{vmatrix} 1 & 7 \\ 2 & 5 \end{vmatrix} - (-2) \begin{vmatrix} 2 & 7 \\ -3 & 5 \end{vmatrix} + (-4) \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} = 1 \cdot (-9) + 2 \cdot 31 - 4 \cdot 7 = 25,$$

where we used the 2×2 determinant formula to calculate each of the three 2×2 determinants. Alternatively, expanding across the second column gives us the same answer,

$$\det(A) = -(-2) \begin{vmatrix} 2 & 7 \\ -3 & 5 \end{vmatrix} + 1 \begin{vmatrix} 1 & -4 \\ -3 & 5 \end{vmatrix} - 2 \begin{vmatrix} 1 & -4 \\ 2 & 7 \end{vmatrix} = 2 \cdot 31 + 1 \cdot (-7) - 2 \cdot 15 = 25.$$

If some of the entries of a matrix A are zero, then it is often easier to calculate $\det(A)$ by expanding across a row or column with the most zeros. For example, suppose

$$A = \begin{bmatrix} 3 & 1 & 2 & 5 \\ 0 & 0 & 2 & 4 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 7 & 8 \end{bmatrix}.$$

Expanding across the first row would lead to four 3×3 determinants. However, expanding across the first column (or third row) would only require one 3×3 determinant calculation because the others would be multiplied by zero. Using the first column, we get

$$\det(A) = 3 \begin{vmatrix} 0 & 2 & 4 \\ 6 & 0 & 0 \\ 0 & 7 & 8 \end{vmatrix} = 3(-6) \begin{vmatrix} 2 & 4 \\ 7 & 8 \end{vmatrix} = 3(-6)(-12) = 216,$$

where the 3×3 determinant was found by expanding across the first column (or equivalently the second row).

10 Cramer's Rule

Determinants can be used to solve systems of equations of the form $A\mathbf{x} = \mathbf{b}$, assuming A is square and $\det(A) \neq 0$. If we let $A_i(\mathbf{b})$ denote matrix A but with column i replaced by $\mathbf{b}$, then according to **Cramer's rule**,

$$x_i = \frac{|A_i(\mathbf{b})|}{|A|} \text{ for } i = 1, 2, \ldots, n.$$

For example, the system of equations

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 4x_2 &= 6 \end{aligned}$$

can be written in the form $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

By Cramer's rule, the solution is

$$x_1 = \frac{|A_1(\mathbf{b})|}{|A|} = \frac{\begin{vmatrix} 5 & 2 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{8}{-2} = -4 \quad \text{and} \quad x_2 = \frac{|A_2(\mathbf{b})|}{|A|} = \frac{\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{-9}{-2} = \frac{9}{2}.$$

11 Inverses

Let A be an $n \times n$ matrix. An $n \times n$ matrix B is called an **inverse** of A if $AB = BA = I$. A is said to be **nonsingular** if $\det(A) \neq 0$ and is said to be **singular** if $\det(A) = 0$. A has an inverse if and only if A is nonsingular. If A has an inverse, then it is unique and it is denoted $B = A^{-1}$.

The inverse of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $\det(A) \neq 0$, is given by the formula

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

There are various algorithms from linear algebra, such as the **cofactor method** or the **Gauss-Jordan method**, for finding inverses of larger matrices.

12 Eigenvalues and Eigenvectors

If A is a square matrix, then the number λ (lambda) is called an **eigenvalue** of A if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Such a vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

Eigenvalues of A are found by solving the polynomial equation

$$\det(A - \lambda I) = 0,$$

called the **characteristic equation**. If A is $n \times n$, then $\det(A - \lambda I)$ is a polynomial of degree n in the variable λ. The eigenvalues of A are the roots of this polynomial. The roots can be real or complex and there may be repeated roots. There will always be n roots, counting multiplicities. If all the entries of A are real, then any complex roots will appear in complex conjugate pairs.

Consider the following example. Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}.$$

To find the eigenvalues of A we compute

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} \\ &= (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 4 + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3). \end{aligned}$$

By solving $\det(A - \lambda I) = 0$ we find that A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$.

For a second example, let

$$A = \begin{bmatrix} -4 & 0 & 3 \\ 0 & 2 & 0 \\ -6 & 0 & 5 \end{bmatrix}.$$

Here we get

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -4-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ -6 & 0 & 5-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} -4-\lambda & 3 \\ -6 & 5-\lambda \end{vmatrix} \\ &= (2-\lambda)[(-4-\lambda)(5-\lambda) + 18] = -(\lambda - 2)(\lambda^2 - \lambda - 20 + 18) \\ &= -(\lambda - 2)(\lambda^2 - \lambda - 2) = -(\lambda - 2)(\lambda - 2)(\lambda + 1) = -(\lambda - 2)^2(\lambda + 1). \end{aligned}$$

In this case $\lambda_1 = 2$ and $\lambda_2 = -1$ are eigenvalues, with $\lambda_1 = 2$ being a repeated root of multiplicity 2.

For a third example, let

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}.$$

In this case,

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5.$$

The roots of this quadratic polynomial are the complex eigenvalues $\lambda = 1 \pm 2i$.

Once the eigenvalues of a matrix are found, then to find the eigenvectors associated with each eigenvalue, we need to solve the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for nonzero vectors $\mathbf{x}$.

For example, to find an eigenvector associated with the eigenvalue $\lambda_1 = 2$ of

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix},$$

we compute

$$A - 2I = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix},$$

and solve

$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This is equivalent to the system of equations

$$\begin{aligned} -x_1 + 2x_2 &= 0 \\ -x_1 + 2x_2 &= 0 \end{aligned}.$$

This system of equations has infinitely many solutions satisfying $x_1 = 2x_2$. The variable x_2 is a "free" variable; it can be chosen to be any value except zero. Note that if x_2 were zero, then x_1 would also be zero, leading to $\mathbf{x} = \mathbf{0}$; but eigenvectors must be nonzero. If we let $x_2 = 1$, then $x_1 = 2$, leading to the eigenvector

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Other choices of x_2 would lead to scalar multiples of this vector, which are also eigenvectors of A associated with $\lambda_1 = 2$.

In general, to solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$, we would use linear algebra techniques such as row-reducing the augmented matrix $[A - \lambda I|\mathbf{0}]$ using the **Gauss-Jordan elimination method**, from which we would obtain the eigenvector solutions.