Linear Systems

A first-order system of linear differential equations has the form

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3 + \cdots + a_{1n}(t)x_n + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + a_{23}(t)x_3 + \cdots + a_{2n}(t)x_n + f_2(t) \\
\frac{dx_3}{dt} &= a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3 + \cdots + a_{3n}(t)x_n + f_3(t) \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + a_{n3}(t)x_3 + \cdots + a_{nn}(t)x_n + f_n(t).
\end{aligned}$$

The system is said to be **homogeneous** if $f_i(t) = 0$ for $i = 1, 2, \ldots, n$; otherwise, it is called **nonhomogeneous**.

If we let

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) & \cdots & a_{2n}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) & \cdots & a_{3n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & a_{n3}(t) & \cdots & a_{nn}(t) \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

where $A(t)$ is the matrix of coefficients, then the system can be written in the matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) & \cdots & a_{2n}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) & \cdots & a_{3n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & a_{n3}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

or simply

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} \quad (\text{or } \mathbf{x}' = A\mathbf{x} \text{ if homogeneous}).$$

We are interested in solving such systems. If subject to an initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0,$$

then we get an initial-value problem.

Theorem 8.1.1 (Existence and Uniqueness of Solutions of Linear IVPs)
If the entries of $A(t)$ and $\mathbf{f}(t)$ are continuous on an interval I containing t_0, then there exists a unique solution of the initial-value problem $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, $\mathbf{x}(t_0) = \mathbf{x}_0$ on I.

Theorem 8.1.2 (Superposition Principle)
If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k$ are solution vectors of the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on an interval I, then the linear combination

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + \cdots + c_k\mathbf{x}_k,$$

where $c_1, c_2, c_3, \ldots, c_k$ are arbitrary constants, is also a solution of the system on I.

Definition 8.1.2 (Linear Dependence/Independence)
A set of solution vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k$ of the homogeneous system $\mathbf{x}' = A\mathbf{x}$ is said to be **linearly dependent** (LD) on an interval I if there exist constants $c_1, c_2, c_3, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + \cdots + c_k\mathbf{x}_k = \mathbf{0},$$

for ever t in I. If the set of vectors is not linearly dependent on I, it is said to be **linearly independent** (LI), in which case $c_1 = c_2 = c_3 = \cdots = c_k = 0$.

Theorem 8.1.3 (Criterion for Linearly Independent Solutions)
Let

$$\mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} x_{13} \\ x_{23} \\ \vdots \\ x_{n3} \end{bmatrix}, \ \ldots, \ \mathbf{x}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

be n solution vectors of the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on an interval I. Then the set of solution vectors is linearly independent on I if and only if the **Wronskian** (an $n \times n$ determinant)

$$W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n) = \begin{vmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{vmatrix} \neq 0,$$

for every t in I.

Definition 8.1.3 (Fundamental Set of Solutions)
Any set $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n$ of n linearly independent solution vectors of the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on an interval I is said to be a **fundamental set of solutions** on I.

Theorem 8.1.4 (Existence of a Fundamental Set)
There exists a fundamental set of solutions for the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on an interval I.

Theorem 8.1.5 (General Solution – Homogeneous Systems)
Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n$ be a fundamental set of solutions of the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on an interval I. Then the **general solution** of the system on I is

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + \cdots + c_n\mathbf{x}_n,$$

where $c_1, c_2, c_3, \ldots, c_n$ are arbitrary constants.

Theorem 8.1.6 (General Solution – Nonhomogeneous Systems)
If $\mathbf{x}_p$ is a particular solution of the nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ on an interval I and $\mathbf{x}_h$ is the general solution of the associated homogeneous system $\mathbf{x}' = A\mathbf{x}$ on I, then the general solution of the nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ on I is given by

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p.$$

Definition (Fundamental Matrix)
If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n$ is a fundamental set of solutions of the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on an interval I, then the matrix

$$\Phi(t) = \left[\mathbf{x}_1 \middle| \mathbf{x}_2 \middle| \mathbf{x}_3 \middle| \cdots \middle| \mathbf{x}_n\right] = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{bmatrix}$$

is called a **fundamental matrix** of the system on I. Its determinant is the Wronskian.

Theorem (Variation of Parameters)
If $\Phi(t)$ is a fundamental matrix of the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on an interval I, then a particular solution of the nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ on I is given by

$$\mathbf{x}_p = \Phi(t) \int \Phi^{-1}(t)\mathbf{f}(t)\, dt.$$

Moreover, since the general solution of the homogeneous system $\mathbf{x}' = A\mathbf{x}$ can be written $\mathbf{x}_h = \Phi(t)\mathbf{c}$ for an arbitrary $n \times 1$ constant vector $\mathbf{c}$, then the general solution of the nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ is given by

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p = \Phi(t)\mathbf{c} + \Phi(t) \int \Phi^{-1}(t)\mathbf{f}(t)\, dt.$$