Linear Differential Equations

A general **n^{th}-order linear differential equation** has the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \tag{1}$$

where $a_n(x)$ is not identically zero. If $g(x)$ is identically zero, the linear differential equation is said to be **homogeneous**; otherwise it is **nonhomogeneous**.

A first-order linear differential equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

which can be converted into an equation of the form

$$\frac{dy}{dx} + P(x)y = f(x),$$

by dividing through by $a_1(x)$, assuming $a_1(x) \neq 0$ for all x in some interval I. The general solution of this equation, which can be found using the integrating factor $\mu(x) = e^{\int P(x)\,dx}$, will involve an arbitrary constant C, which can be solved for if given an initial condition $y(x_0) = y_0$.

The general solution of an n^{th}-order linear differential equation will involve n arbitrary constants $c_1, c_2, \ldots, c_n$, which can be solved for given n initial conditions at x_0:

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_2, \ldots,\ y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

These initial conditions (2) together with the differential equation (1) define an **initial-value problem** (IVP). The following theorem guarantees the existence of a unique solution of the IVP.

Theorem 4.1.1 (Existence and Uniqueness of Solutions of Linear IVPs)
Let $a_n(x), a_{n-1}(x), \ldots, a_2(x), a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for all x in I. If $x = x_0$ is any point in I, then there exists a unique solution $y(x)$ of the initial-value problem (1) and (2) on I.

Note that Theorem 4.1.1 guarantees the existence of a unique solution on the *entire* interval I. This is in contrast to Theorem 1.2.1, which guaranteed the existence of a unique solution to the first-order nonlinear initial-value problem $dy/dx = f(x, y)$ subject to $y(x_0) = y_0$, where under suitable continuity assumptions on f and $\partial f/\partial y$, a unique solution was only guaranteed to exist in some possibly very tiny interval around x_0.

If $a_n(x) \neq 0$ for all x in I, then the differential equation (1) can be simplified by dividing through by $a_n(x)$ as was done in the case of the first-order linear differential equation. For example, a second-order linear IVP would become

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = y_0,\ y'(x_0) = y_1$$

Theorem 4.1.2 (Superposition Principle – Homogeneous Equations)
Let $y_1, y_2, y_3, \ldots, y_k$ be solutions of the homogeneous n^{th}-order linear differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{3}$$

on an interval I. Then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + c_3y_3(x) + \cdots + c_ky_k(x),$$

where $c_1, c_2, c_3, \ldots, c_k$ are arbitrary constants, is also a solution on the interval I.

Definition 4.1.1 (Linear Dependence/Independence)
A set of functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ is said to be **linearly dependent** (LD) on an interval I if there exist constants $c_1, c_2, c_3, \ldots, c_n$, not all zero, such that

$$c_1f_1(x) + c_2f_2(x) + c_3f_3(x) + \cdots + c_nf_n(x) = 0 \tag{4}$$

for every x in the interval I. If the set of functions is not linearly dependent on I, it is said to be **linearly independent** (LI), in which case the only constants for which (4) is satisfied for every x in I are $c_1 = c_2 = c_3 = \cdots = c_n = 0$.

Definition 4.1.2 (Wronskian)
Suppose each of the functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, f_3, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1' & f_2' & f_3' & \cdots & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions.

Theorem 4.1.3 (Criterion for Linearly Independent Solutions)
Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous n^{th}-order linear differential equation (3) on an interval I. Then the set of solutions is linearly independent on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in I.

Definition 4.1.3 (Fundamental Set of Solutions)
Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous n^{th}-order linear differential equation (3) on an interval I is said to be a **fundamental set of solutions** on I.

Theorem 4.1.4 (Existence of a Fundamental Set)
There exists a fundamental set of solutions for the homogeneous n^{th}-order linear differential equation (3) on an interval I.

Theorem 4.1.5 (General Solution – Homogeneous Equations)
Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous n^{th}-order linear differential equation (3) on an interval I. Then the **general solution** of the equation on I is

$$y = c_1y_1(x) + c_2y_2(x) + c_3y_3(x) + \cdots + c_ny_n(x),$$

where $c_1, c_2, c_3, \ldots, c_n$ are arbitrary constants.