C A M O S U N C O L L E G E

MATHEMATICS 226

Fourier Series

Fourier Series

There is the need in applied mathematics for general periodic functions. They are required, for example, in finding solutions to partial differential equations. Instead of using power series, it is more appropriate to expand periodic functions in terms of sines and cosines. Infinite series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$$

are called Fourier series. The functions $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$ are periodic with period $T = \frac{2L}{n}$. To see this, simply use identities to write out

$$\cos\frac{n\pi(x+T)}{L} = \cos\frac{n\pi x}{L}\cos\frac{n\pi T}{L} - \sin\frac{n\pi x}{L}\sin\frac{n\pi T}{L}.$$

This can only be equal to $\cos\frac{n\pi x}{L}$ if $\cos\frac{n\pi T}{L} = 1$ and $\sin\frac{n\pi T}{L} = 0$. The only solutions to these equations are for

$$\frac{n\pi T}{L} = 2k\pi \text{ for any integer } k.$$

The fundamental period of $\cos\frac{n\pi x}{L}$ is when $k = 1$ so $\frac{n\pi T}{L} = 2\pi$ and hence $T = \frac{2L}{n}$. The same can be shown for $\sin\frac{n\pi x}{L}$.

As with power series, the problem is to compute the infinitely many coefficients a_n and b_n. With power series, we differentiate. With Fourier series, we integrate. Assuming we can integrate term by term, consider

$$\begin{aligned}
\int_{-L}^{L} f(x)dx &= \int_{-L}^{L}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})\right]dx \\
&= \frac{a_0}{2}\int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L}\cos\frac{n\pi x}{L}dx + \sum_{n=1}^{\infty} b_n \int_{-L}^{L}\sin\frac{n\pi x}{L}dx \\
&= \frac{a_0}{2}\int_{-L}^{L} dx \\
&= a_0 L
\end{aligned}$$

where we have used that the integrals of the sine and cosine functions over multiples of their periods are zero. So we have the first coefficient

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx.$$

Note that the first term $\frac{a_0}{2}$ of the series is just the average value of $f(x)$ on the interval $[-L, L]$. In order to compute the rest of the coefficients, we will need to use the idenities

$$\begin{aligned}
\cos A \cos B &= \tfrac{1}{2}[\cos(A-B) + \cos(A+B)] \\
\cos A \sin B &= \tfrac{1}{2}[\sin(A+B) - \sin(A-B)] \\
\sin A \sin B &= \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]
\end{aligned}$$

to the compute the following integrals:

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0 \text{ if } m \neq n \\ L \text{ if } m = n \end{cases}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = 0$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} 0 \text{ if } m \neq n \\ L \text{ if } m = n \end{cases}$$

The last one for example, if $m \neq n$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L} \cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}dx$$

$$= \frac{L}{2\pi}\left[\frac{\sin\frac{(m-n)\pi x}{L}}{m-n} - \frac{\sin\frac{(m+n)\pi x}{L}}{m+n}\right]\Bigg|_{-L}^{L} = 0 .$$

If $m = n$, then

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{n\pi x}{L}dx = \int_{-L}^{L} \sin^2\frac{n\pi x}{L}dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left[1 - \cos\frac{2n\pi x}{L}\right]dx$$

$$= \frac{1}{2}\left[x - \frac{\sin\frac{2n\pi x}{L}}{\frac{2n\pi}{L}}\right]\Bigg|_{-L}^{L} = L$$

With these integrals, we can compute the rest of the Fourier coefficients. Now we multiply by $\cos\frac{n\pi x}{L}$ and integrate.

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \int_{-L}^{L}\left[\frac{a_0}{2}\cos\frac{n\pi x}{L} + \sum_{m=1}^{\infty}(a_m\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} + b_m\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L})\right]dx$$

$$= 0 + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx + 0 = a_n L$$

and so we have

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx , \; n = 1, 2, 3, ...$$

and similarly, multiplying by $\sin\frac{n\pi x}{L}$ and integrating,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx , \; n = 1, 2, 3, ...$$

Fourier Series

Example 1: Find the Fourier series for the function $f(x) = |x|$ on $[-L, L]$.

Using the formulae above

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= -\frac{1}{L}\int_{-L}^{0} xdx + \frac{1}{L}\int_{0}^{L} xdx \\ &= \frac{2}{L}\int_{0}^{L} xdx \\ &= L \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \\ &= -\frac{1}{L}\int_{-L}^{0} x\cos\frac{n\pi x}{L}dx + \frac{1}{L}\int_{0}^{L} x\cos\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\int_{0}^{L} x\cos\frac{n\pi x}{L}dx \\ &= \frac{2L}{n^2\pi^2}(\cos n\pi - 1) \\ &= \begin{cases} 0 & \text{if } n \text{ even} \\ -\frac{4L}{n^2\pi^2} & \text{if } n \text{ odd} \end{cases} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \\ &= -\frac{1}{L}\int_{-L}^{0} x\sin\frac{n\pi x}{L}dx + \frac{1}{L}\int_{0}^{L} x\sin\frac{n\pi x}{L}dx \\ &= 0 \end{aligned}$$

Thus the series is

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}) &= \frac{L}{2} - \frac{4L}{\pi^2}\Big[\cos\frac{\pi x}{L} + \frac{1}{3^2}\cos\frac{3\pi x}{L} + \frac{1}{5^2}\cos\frac{5\pi x}{L}...\Big] \\ &= \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L} \end{aligned}$$

One may now ask: does this series converge, and if so, does it converge to the original $f(x)$? The answer is given by the following theorem.

The Fourier Theorem

Suppose that $f(x)$ and $f'(x)$ are piecewise continuous on $[-L, L]$ and $f(x)$ is defined outside of $[-L, L]$ as being periodic with period $2L$. Then $f(x)$ has a Fouier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L})$$

and the series converges to $f(x)$ wherever $f(x)$ is continuous and to the average of the left and right limits at points where $f(x)$ is discontinuous.

Fourier Series

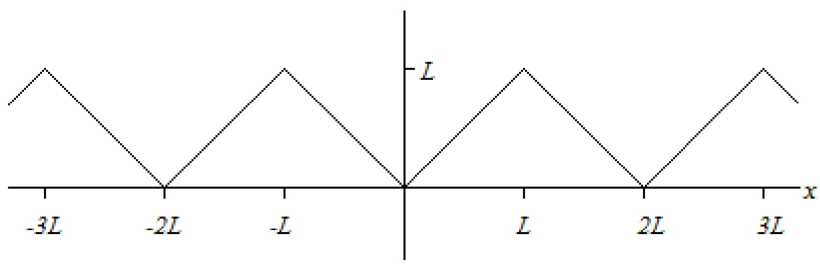
It is common to use the notation for the left and right hand limits as $f(a^-) = \lim_{x \to a^-} f(x)$ and $f(a^+) = \lim_{x \to a^+} f(x)$. With this notation, one can say the the Fourier series converges at a jump (actually anywhere) to

$$\frac{f(x^-) + f(x^+)}{2}.$$

By the theorem, we have for the above example that the series does converge and that

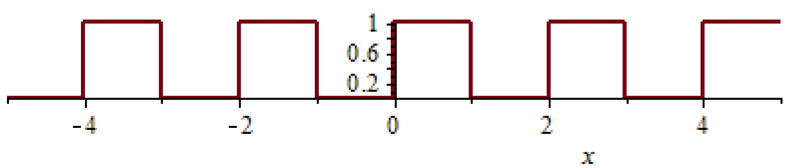
$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}$$

is a periodic function defined for all x which gives $|x|$ on $[-L, L]$ and, what is called the periodic extension of $|x|$ everywhere else. This is a saw tooth wave.



Next we will find the Fourier series for a square wave.

Example 2: Let $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } \ \ 0 < x < 1 \end{cases}$ and suppose this is defined everywhere else by periodic extension.



Now this is a odd function (shifted up ½) and so the series will have only sine terms ($a_n = 0$ for all $n \geq 1$) and

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^{L} f(x)dx \\ &= \int_{-1}^{1} f(x)dx \\ &= \int_{0}^{1} dx \\ &= 1 \end{aligned}$$

Fourier Series

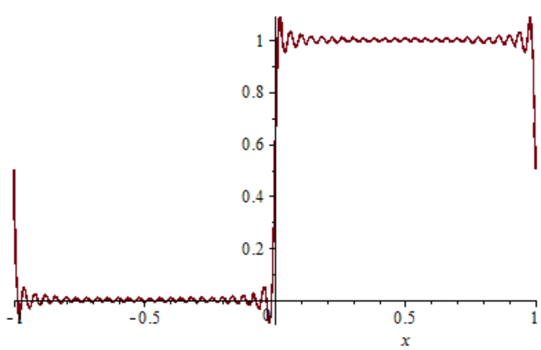
$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \\
&= \int_{-1}^{1} f(x)\sin n\pi x dx \\
&= \int_{-1}^{0} 0\sin n\pi x dx + \int_{0}^{1} 1\sin n\pi x dx \\
&= \frac{1}{n\pi}(1-\cos n\pi) \\
&= \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{2}{n\pi} & \text{if } n \text{ odd} \end{cases}
\end{aligned}
$$

and so

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin n\pi x \\
&= \frac{1}{2} + \frac{2}{\pi}\left[\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5^2}\sin 5\pi x \ldots\right] \\
&= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)}\sin((2n-1)\pi x)
\end{aligned}
$$

It is interesting to see how sine waves are used to approximate this square wave. The graph below shows the sum of the terms to $n = 25$ of the Fourier series. Notice how the series is converging to the half way points where the original function jumps. Note also the behavior of the Fourier series near the corners of the square wave. These "over shoots" are called Gibb's phenomenon and are responsible for "ringing" in electrical circuits.

$n = 25$



Fourier Series

Exercises

1. Find the Fourier series for the periodic extension of $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } \ \ 0 < x < 1 \end{cases}$. What does the series converge to when $x = 1$?

2. Find the Fourier series for $\cos^2 2x$.

3. Sketch the periodic extension of $f(x) = x^2$ on $[0, 2]$. Show that the Fourier series for this function is

$$f(x) = \frac{4}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi x.$$

By computing $f(0)$, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}.$$