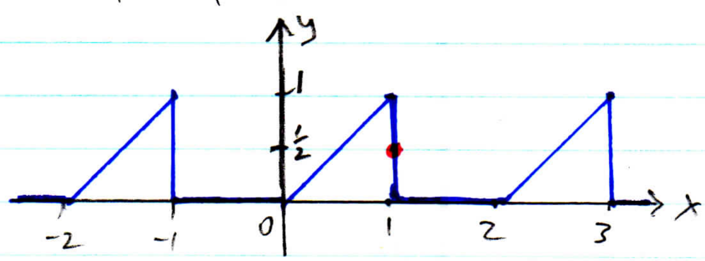
# Fourier Series – Exercise Solutions

1. $f(x) = \begin{cases} 0, & \text{if } -1<x<0 \\ x, & \text{if } 0<x<1 \end{cases}$

$\therefore L=1$ (period 2)

Graph of periodic extension:




$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_0^1 x\,dx = \tfrac{1}{2}x^2\Big|_0^1 = 1/2$$

For $n \geq 1$, $a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}\,dx = \int_0^1 x\cos n\pi x\,dx$

Int. by parts:

| | u | v′ |
|---|---|---|
| + | $x$ | $\cos n\pi x$ |
| − | $1$ | $\frac{1}{n\pi}\sin n\pi x$ |
| + | $0$ | $\frac{-1}{n^2\pi^2}\cos n\pi x$ |

$$= \left[\frac{1}{n\pi}x\sin n\pi x + \frac{1}{n^2\pi^2}\cos n\pi x\right]_0^1$$

$$= \frac{1}{n^2\pi^2}(\cos n\pi - 1) = \frac{(-1)^n - 1}{n^2\pi^2}$$

$$= \begin{cases} 0, & \text{if } n \text{ even} \\ \frac{-2}{n^2\pi^2}, & \text{if } n \text{ odd} \end{cases}$$

For $n \geq 1$, $b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{1}\,dx = \int_0^1 x\sin n\pi x\,dx$

Int. by parts:

| | u | v′ |
|---|---|---|
| + | $x$ | $\sin n\pi x$ |
| − | $1$ | $-\frac{1}{n\pi}\cos n\pi x$ |
| + | $0$ | $-\frac{1}{n^2\pi^2}\sin n\pi x$ |

$$= \left[-\frac{1}{n\pi}x\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]_0^1$$

$$= \frac{-1}{n\pi}\cos n\pi = \frac{(-1)^{n+1}}{n\pi}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{1} + b_n\sin\frac{n\pi x}{1}\right)$$

$$= \frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x + \frac{(-1)^{n+1}}{n\pi}\sin n\pi x\right)$$

$$= \frac{1}{4} + \underbrace{\sum_{n=1}^{\infty}\frac{-2}{(2n-1)^2\pi^2}\cos(2n-1)\pi x}_{\text{odd terms}} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}\sin n\pi x$$

At $x=1$, series converges to $\dfrac{f(1^+)+f(1^-)}{2} = \dfrac{0+1}{2} = \dfrac{1}{2}$ (● on graph) [note: $f(1^+) = f(-1^+)$]

2. $f(x) = \cos^2 2x = \dfrac{1+\cos 4x}{2} = \underbrace{\tfrac{1}{2} + \tfrac{1}{2}\cos 4x}$

This is the Fourier series on $[-\pi/4, \pi/4]$, which is one period of $f$. Here $L = \pi/4$.

Let's check. $a_0 = \dfrac{1}{\pi/4}\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right) dx$

$$= \frac{4}{\pi}\left[\tfrac{1}{2}x + \tfrac{1}{8}\sin 4x\right]_{-\pi/4}^{\pi/4} = \frac{4}{\pi}\left(\frac{\pi}{8} + \frac{\pi}{8}\right) = 1$$

For $n \geq 1$, $a_n = \dfrac{1}{\pi/4}\displaystyle\int_{-\pi/4}^{\frac{\pi}{4}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right)\cos\frac{n\pi x}{\pi/4}\, dx$

$$= 4/\pi \int_{-\pi/4}^{\pi/4} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right)\cos 4nx\, dx$$

$$= \frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\pi/4} \left(\cos 4nx + \cos 4x \cos 4nx\right) dx$$

$$= \frac{2}{\pi}\left[\cancelto{0}{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 4nx\, dx} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 4x \cos 4nx\, dx\right]$$

$$= \frac{2}{\pi} \cdot \begin{cases} \frac{\pi}{4} & \text{if } n=1 \\ 0 & \text{if } n \neq 1 \end{cases} = \begin{cases} \frac{1}{2} & \text{if } n=1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

For $n \geq 1$, $b_n = \dfrac{1}{\pi/4}\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right)\sin\frac{n\pi x}{\pi/4}\, dx$

$$= \frac{4}{\pi}\int_{-\frac{\pi}{4}}^{\pi/4} \underbrace{\underbrace{\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right)}_{\text{even}} \underbrace{\sin 4nx}_{\text{odd}}}_{\text{odd}}\, dx = 0$$

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{\pi/4} + \cancelto{0}{b_n \sin\frac{n\pi x}{\pi/4}}\right]$$

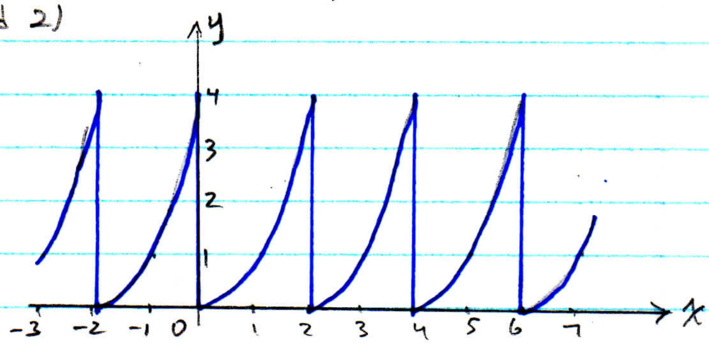
$$= \frac{1}{2} + \frac{1}{2}\cos 4x$$

$\uparrow$

$a_1$ $\quad (a_2 = a_3 = a_4 = \cdots = 0)$

3. $f(x) = x^2$ on $[0,2]$ Graph of periodic extension:

$\therefore L = 1$ (period 2)




Easier to find $a_n$ and $b_n$ by integrating over the period $[0,2]$ instead of $[-1,1]$.

$$a_0 = \frac{1}{1}\int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = 8/3$$

For $n \geq 1$, $a_n = \frac{1}{1}\int_0^2 x^2 \cos n\pi x\,dx = \left[\frac{1}{n\pi}x^2 \sin n\pi x + \frac{2}{n^2\pi^2}x\cos n\pi x - \frac{2}{n^3\pi^3}\sin n\pi x\right]_0^2$

$= \frac{4}{n^2\pi^2}$

Int by parts:

| | $u$ | $v'$ |
|---|---|---|
| $+$ | $x^2$ | $\cos n\pi x$ |
| $-$ | $2x$ | $\frac{1}{n\pi}\sin n\pi x$ |
| $+$ | $2$ | $-\frac{1}{n^2\pi^2}\cos n\pi x$ |
| $-$ | $0$ | $-\frac{1}{n^3\pi^3}\sin n\pi x$ |

For $n \geq 1$, $b_n = \frac{1}{1}\int_0^2 x^2 \sin n\pi x\,dx = \left[-\frac{1}{n\pi}x^2\cos n\pi x + \frac{2}{n^2\pi^2}x\sin n\pi x + \frac{2}{n^3\pi^3}\cos n\pi x\right]_0^2$

$= \left(\frac{-4}{n\pi} + \frac{2}{n^3\pi^3}\right) - \left(\frac{2}{n^3\pi^3}\right) = \frac{-4}{n\pi}$

Int by parts:

| | $u$ | $v'$ |
|---|---|---|
| $+$ | $x^2$ | $\sin n\pi x$ |
| $-$ | $2x$ | $-\frac{1}{n\pi}\cos n\pi x$ |
| $+$ | $2$ | $-\frac{1}{n^2\pi^2}\sin n\pi x$ |
| $-$ | $0$ | $\frac{1}{n^3\pi^3}\cos n\pi x$ |

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{1} + b_n \sin\frac{n\pi x}{1}\right) = \frac{4}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}\cos n\pi x - \frac{4}{n\pi}\sin n\pi x\right)$$

$$= \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x.$$

For periodic extension, $f(0) = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\cdot 1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cdot 0$

$$= \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Meanwhile, $f(0) = \frac{f(0^-)+f(0^+)}{2} = \frac{4+0}{2} = 2$

$$\therefore \quad \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = 2$$

$$\Rightarrow \quad \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$