Convolution

Consider the problem of solving the nonhomogeneous second-order linear differential equation

$$x'' + \omega^2 x = f(t),$$

where $\omega > 0$ and $f(t)$ is a continuous function on $[0, \infty)$. The solution will be $x(t) = x_h + x_p$, where the homogeneous part of the solution is

$$x_h = c_1 \cos \omega t + c_2 \sin \omega t,$$

and variation of parameters can be used to find a particular solution of the form

$$x_p = u_1 x_1 + u_2 x_2 = u_1 \cos \omega t + u_2 \sin \omega t.$$

The Wronskian is

$$W = \begin{vmatrix} \cos \omega t & \sin \omega t \\ -\omega \sin \omega t & \omega \cos \omega t \end{vmatrix} = \omega \cos^2 \omega t + \omega \sin^2 \omega t = \omega,$$

from which we get

$$u_1' = \frac{\begin{vmatrix} 0 & \sin \omega t \\ f(t) & \omega \cos \omega t \end{vmatrix}}{\omega} = -\frac{f(t) \sin \omega t}{\omega} \quad \text{and} \quad u_2' = \frac{\begin{vmatrix} \cos \omega t & 0 \\ -\omega \sin \omega t & f(t) \end{vmatrix}}{\omega} = \frac{f(t) \cos \omega t}{\omega}.$$

Integrating u_1' to get u_1 gives

$$u_1 = -\frac{1}{\omega} \int f(t) \sin \omega t \, dt,$$

where the integration constant can be any number since any antiderivative will do. Instead of writing u_1 as an indefinite integral, it can instead be written as a definite integral function of t of the form

$$u_1 = -\frac{1}{\omega} \int_0^t f(\tau) \sin \omega \tau \, d\tau,$$

where τ is used as a "dummy variable" inside the integral. Similarly, u_2 can be written

$$u_2 = \frac{1}{\omega} \int_0^t f(\tau) \cos \omega \tau \, d\tau,$$

A particular solution is therefore

$$\begin{aligned}
x_p &= \left(-\frac{1}{\omega} \int_0^t f(\tau) \sin \omega \tau \, d\tau \right) \cos \omega t + \left(\frac{1}{\omega} \int_0^t f(\tau) \cos \omega \tau \, d\tau \right) \sin \omega t \\
&= -\frac{1}{\omega} \int_0^t f(\tau) \sin \omega \tau \cos \omega t \, d\tau + \frac{1}{\omega} \int_0^t f(\tau) \cos \omega \tau \sin \omega t \, d\tau \\
&= -\frac{1}{\omega} \left(\int_0^t f(\tau) \sin \omega \tau \cos \omega t \, d\tau - \int_0^t f(\tau) \cos \omega \tau \sin \omega t \, d\tau \right) \\
&= -\frac{1}{\omega} \int_0^t f(\tau) (\sin \omega \tau \cos \omega t - \cos \omega \tau \sin \omega t) \, d\tau = -\frac{1}{\omega} \int_0^t f(\tau) \sin(\omega \tau - \omega t) \, d\tau \\
&= -\frac{1}{\omega} \int_0^t f(\tau) \sin \omega (\tau - t) \, d\tau = \frac{1}{\omega} \int_0^t f(\tau) \sin \omega (t - \tau) \, d\tau.
\end{aligned}$$

This particular solution, which is expressed as an integral in terms of $f(t)$ and $\sin \omega t$, is an example of what is called a **convolution** of two functions.

Definition: If $f(t)$ and $g(t)$ are piecewise continuous functions on $[0, \infty)$, then the **convolution** of f and g, denoted $f * g$, is defined by

$$f * g = \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

The convolution may be written $(f * g)(t)$ to emphasize that it is a function of t. □

Convolution Theorem: If $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$, then

$$\mathscr{L}\{f * g\} = F(s)G(s), \quad \text{or equivalently,} \quad \mathscr{L}^{-1}\{F(s)G(s)\} = f * g.$$ □

We can use Laplace transforms to solve the initial-value problem

$$x'' + \omega^2 x = f(t), \quad x(0) = A, x'(0) = B.$$

Transforming the equation gives

$$\begin{aligned}
\mathscr{L}\{x''\} + \omega^2 \mathscr{L}\{x\} &= \mathscr{L}\{f(t)\} \\
s^2 X(s) - s x(0) - x'(0) + \omega^2 X(s) &= F(s) \\
s^2 X(s) - As - B + \omega^2 X(s) &= F(s) \\
(s^2 + \omega^2) X(s) &= As + B + F(s) \\
X(s) &= \frac{As}{s^2 + \omega^2} + \frac{B}{s^2 + \omega^2} + \frac{F(s)}{s^2 + \omega^2} \\
X(s) &= A \frac{s}{s^2 + \omega^2} + \frac{B}{\omega} \frac{\omega}{s^2 + \omega^2} + \frac{1}{\omega} F(s) \frac{\omega}{s^2 + \omega^2}.
\end{aligned}$$

Now taking inverse Laplace transforms and applying the convolution theorem to the third term, we get the general solution of the nonhomogeneous equation.

$$\begin{aligned}
x(t) &= A \mathscr{L}^{-1}\left\{\frac{s}{s^2 + \omega^2}\right\} + \frac{B}{\omega} \mathscr{L}^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\} + \frac{1}{\omega} \mathscr{L}^{-1}\left\{F(s) \frac{\omega}{s^2 + \omega^2}\right\} \\
&= A \cos \omega t + \frac{B}{\omega} \sin \omega t + \frac{1}{\omega}(f(t) * \sin \omega t) \\
&= A \cos \omega t + \frac{B}{\omega} \sin \omega t + \frac{1}{\omega} \int_0^t f(\tau) \sin \omega (t - \tau)\, d\tau.
\end{aligned}$$