Strong Induction

Let $P(n)$ be a propositional function of an integer n.

Recall that the **Principle of Mathematical Induction** (PMI) establishes the validity of the following "argument".

$$
\begin{array}{ll}
 & P(1) \\
 & P(1) \rightarrow P(2) \\
 & P(2) \rightarrow P(3) \\
 & P(3) \rightarrow P(4) \\
 & \qquad \vdots \\
 & P(k) \rightarrow P(k+1) \\
 & \qquad \vdots \\
\hline
\therefore & P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k) \wedge P(k+1) \wedge \cdots
\end{array}
$$

Another type of induction, known as **Strong Induction** (or the **Second Principle of Mathematical Induction**), states that the following "argument" is valid.

$$
\begin{array}{ll}
 & P(1) \\
 & P(1) \rightarrow P(2) \\
 & [P(1) \wedge P(2)] \rightarrow P(3) \\
 & [P(1) \wedge P(2) \wedge P(3)] \rightarrow P(4) \\
 & [P(1) \wedge P(2) \wedge P(3) \wedge P(4)] \rightarrow P(5) \\
 & \qquad \vdots \\
 & [P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k)] \rightarrow P(k+1) \\
 & \qquad \vdots \\
\hline
\therefore & P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k) \wedge P(k+1) \wedge \cdots
\end{array}
$$

Symbolically, PMI is a rule of inference that asserts

$$[P(1) \wedge \forall k \in \mathbb{Z}^+ (P(k) \rightarrow P(k+1))] \rightarrow \forall n \in \mathbb{Z}^+ P(n),$$

whereas Strong Induction is a rule of inference that asserts

$$[P(1) \wedge \forall k \in \mathbb{Z}^+ ([P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k)] \rightarrow P(k+1))] \rightarrow \forall n \in \mathbb{Z}^+ P(n).$$

To prove $P(n)$ is true for all $n \in \mathbb{Z}^+$ using **Strong Induction**, one must complete two steps:

1. **Base Case:** Show $P(1)$ is true.
2. **Inductive Step:** Show that $[P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for every $k \in \mathbb{Z}^+$.

The base case in Strong Induction, as in PMI, need not start at $n = 1$ but could start at any integer b. In this case in order to prove $P(n)$ is true for all $n \geq b$ one must complete these two steps:

1. **Base Case:** Show $P(b)$ is true.
2. **Inductive Step:** Show that $[P(b) \wedge P(b+1) \wedge P(b+2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for every integer $k \geq b$.

An alternative version of Strong Induction for proving $P(n)$ is true for all $n \geq b$ is to complete these two modified steps, where i represents some integer with $i \geq b$:

1. **Base Case:** Show $P(b), P(b+1), P(b+2), \ldots, P(i)$ are all true.
2. **Inductive Step:** Show that $[P(b) \wedge P(b+1) \wedge P(b+2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for every integer $k \geq i$.

Strong Induction is normally used instead of PMI when, in proving the inductive step, $P(k)$ alone is not enough to show $P(k+1)$ but rather you need to also assume $P(j)$ for one or more j values less than k.