Set Operations

Definition Let A be a set contained in some universal set U. Then the **complement** of A (relative to U), denoted by $\overline{A}$ (or A' or A^c), is defined by

$$\overline{A} = \{x | x \notin A\}.$$

Definition Let A and B be sets contained in some universal set U. Then their **intersection**, denoted by $A \cap B$, is defined by

$$A \cap B = \{x | (x \in A) \wedge (x \in B)\}.$$

Definition Let A and B be sets contained in some universal set U. Then their **union**, denoted by $A \cup B$, is defined by

$$A \cup B = \{x | (x \in A) \vee (x \in B)\}.$$

Recall that $\vee$ is inclusive, so $A \cup B$ consists of all those elements that are either in A or in B or both.

Definition Let A and B be sets contained in some universal set U. Then their **difference** (or **relative complement**), denoted by $A - B$ (or $A \backslash B$), is defined by

$$A - B = \{x | (x \in A) \wedge (x \notin B)\}.$$

In general $A - B \neq B - A$. In fact $A - B = B - A$ if, and only if, $A = B$, in which case $A - B = B - A = \emptyset$.

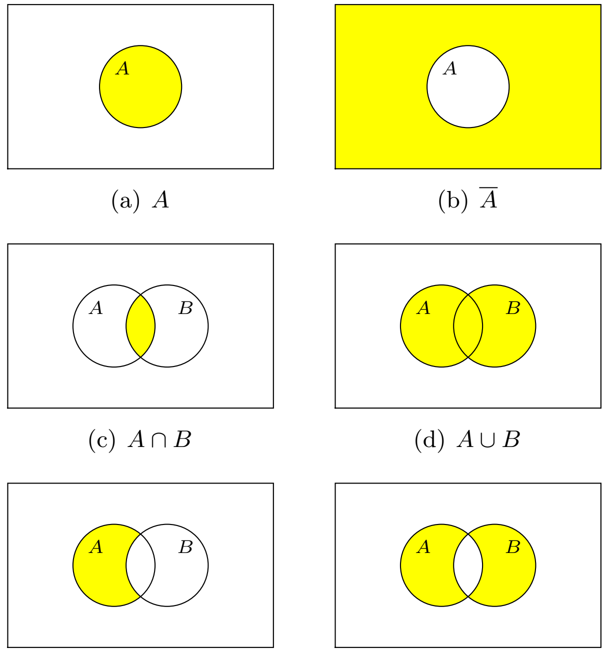
It can be easily proven that $A - B = A \cap \overline{B}$.

Definition Let A and B be sets contained in some universal set U. Then their **symmetric difference**, denoted by $A \oplus B$ (or $A \triangle B$), is defined by

$$A \oplus B = \{x | (x \in A) \oplus (x \in B)\}.$$

The symbol $\oplus$ is used to denote both the exclusive or logic operation as well as the symmetric difference set operation. Note that $A \oplus B = B \oplus A$. It can also be easily proven that

$$A \oplus B = (A - B) \cup (B - A) = (A \cap \overline{B}) \cup (B \cap \overline{A}), \quad \text{and}$$
$$A \oplus B = (A \cup B) - (A \cap B) = (A \cup B) \cap \overline{A \cap B}.$$



(a) A (b) $\overline{A}$

(c) $A \cap B$ (d) $A \cup B$

(e) $A - B$ (f) $A \oplus B$

Figure 1: Venn Diagrams (the rectangle represents the universal set U)

Example: Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 3, 5, 7, 9\}$ and $B = \{3, 4, 5, 6\}$. Then

$$\begin{aligned}
\overline{A} &= \{2, 4, 6, 8, 10\} \\
\overline{B} &= \{1, 2, 7, 8, 9, 10\} \\
A \cap B &= \{3, 5\} \\
A \cup B &= \{1, 3, 4, 5, 6, 7, 9\} \\
A - B &= \{1, 7, 9\} \\
B - A &= \{4, 6\} \\
A \oplus B &= \{1, 4, 6, 7, 9\}
\end{aligned}$$

Note that

$$\begin{aligned}
\overline{A \cup B} &= \overline{\{1, 3, 4, 5, 6, 7, 9\}} = \{2, 8, 10\}, \quad \text{and} \\
\overline{A} \cap \overline{B} &= \{2, 4, 6, 8, 10\} \cap \{1, 2, 7, 8, 9, 10\} = \{2, 8, 10\}.
\end{aligned}$$

It is **not** a coincidence that $\overline{A \cup B} = \overline{A} \cap \overline{B}$ for the given sets A and B. In fact this equality

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

(called De Morgan's law for sets) is true in general for arbitrary sets A and B.

Prove: $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof 1: One way to prove two sets are equal is to show they are subsets of each other. In other words we can prove (i) $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and (ii) $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. In each case, to show that one set is a subset of another we can show that any arbitrary element x that belongs to the first set must also belong to the second set. You will note that in this example, as in many examples that involve proving the equality of sets, the sequence of steps used to prove part (ii) is simply a reversal of the steps used to prove part (i).

(i) Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$ (by definition of complement). Therefore $\neg(x \in (A \cup B))$ is true (by definition of $\notin$) and so $\neg((x \in A) \vee (x \in B))$ is true (by definition of union). By De Morgan's law of logic $\neg(x \in A)$ and $\neg(x \in B)$ and so $x \notin A$ and $x \notin B$ (by definition of $\notin$). Thus $x \in \overline{A}$ and $x \in \overline{B}$ (by definition of complement) and finally $x \in \overline{A} \cap \overline{B}$ (by definition of intersection). This proves $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

(ii) Let $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$ (by definition of intersection). Therefore $x \notin A$ and $x \notin B$ by (by definition of complement) and so $\neg(x \in A)$ and $\neg(x \in B)$ (by definition of $\notin$). By De Morgan's law of logic $\neg((x \in A) \vee (x \in B))$ and so $\neg(x \in (A \cup B))$ (by definition of union). Therefore $x \notin A \cup B$ (by definition of $\notin$) and so $x \in \overline{A \cup B}$ (by definition of complement). This proves $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

Proof 2: A quicker way of proving this set identity is to establish a sequence of equalities.

$$
\begin{aligned}
\overline{A \cup B} &= \{x | x \notin A \cup B\} && \text{by definition of complement} \\
&= \{x | \neg(x \in A \cup B)\} && \text{by definition of } \notin \\
&= \{x | \neg((x \in A) \vee (x \in B))\} && \text{by definition of union} \\
&= \{x | \neg(x \in A) \wedge \neg(x \in B)\} && \text{by De Morgan's law of logic} \\
&= \{x | (x \notin A) \wedge (x \notin B)\} && \text{by definition of } \notin \\
&= \{x | (x \in \overline{A}) \wedge (x \in \overline{B})\} && \text{by definition of complement} \\
&= \overline{A} \cap \overline{B} && \text{by definition of intersection}
\end{aligned}
$$

Note: A Venn Diagram may be useful for checking whether one set is equal to or is a subset of another; however, <u>a Venn Diagram is not considered an acceptable method of proof.</u>