CAMOSUN

Name: SOLUTIONS

MATH 126 (Winter, 2014)
Term Test 3
by George Ballinger

Answer the questions in the space provided. This test has 10 questions for a total of 25 marks. Express answers to counting problems as integers and show your calculations.

1. (6 marks) How many strings of six letters from the alphabet $\{A, B, C, \ldots, Z\}$ are there

   (a) if letters **cannot** be repeated?

   $P(26,6) = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 = 165{,}765{,}600$

   (b) that start with one of the vowels A, E, I, O or U if letters **can** be repeated?

   $5 \cdot 26^5 = 59{,}406{,}880$

   (c) that start with A or end with Z if letters **can** be repeated?

   $\underbrace{1 \cdot 26^5}_{\text{starts with A}} + \underbrace{26^5 \cdot 1}_{\text{ends with Z}} - \underbrace{1 \cdot 26^4 \cdot 1}_{\text{Start with A and ends with Z}} = 23{,}305{,}776$ (P.I.E.)

   (d) that have at least one A if letters **can** be repeated?

   i.e. not 0 A's

   $\underbrace{26^6}_{\text{all strings}} - \underbrace{25^6}_{\text{strings with 0 A's}} = 64{,}775{,}151$

2. (2 marks) A school baseball coach must choose 9 children from among a group of 7 girls and 20 boys to form a team. If the team must have either 2 or 3 girls on it, then in how many ways can the coach form such a team?

$$C(7,2)\cdot C(20,7)+C(7,3)\cdot C(20,6)$$
$$= 1,627,920 + 1,356,600$$
$$= 2,984,520$$

3. (3 marks) How many strings of length 9 are there, using symbols from $\{G, O, D\}$, if

   (a) each string contains exactly 4 G's, 3 O's, and 2 D's?

   GGGGOOODD

   $$\frac{9!}{4!\,3!\,2!} = 1260 \quad \text{OR} \quad C(9,4)\cdot C(5,3)\cdot C(2,2) = 1260$$

   (b) each string contains exactly 4 G's, 3 O's, and 2 D's and the 3 O's are consecutive (as in "OOO")?

   $\underbrace{\text{GGGG}\boxed{\text{OOO}}\text{DD}}_{\text{7 objects}}$ 1 block

   $$\frac{7!}{4!\,1!\,2!} = 105 \quad \text{OR} \quad C(7,4)\cdot C(3,1)\cdot C(2,2) = 105$$

4. (1 mark) What is the minimum number of students required in a discrete mathematics class to be sure that at least three will receive the same grade if there are 10 possible grades, A+, A, A-, B+, B, B-, C+, C, D, and F?

   $10 \cdot 2 + 1 = 21$

   20 is too few since there could be exactly 2 students with each grade.

   Since $\lceil \frac{21}{10} \rceil = 3$ then 21 students are sufficient by generalized pigeonhole principle

5. (1 mark) What is the coefficient of $x^{27}y^3$ in the binomial expansion of $(x+y)^{30}$?

   $$\binom{30}{3} = 4060$$

6. (3 marks) A bowl contains a large mixture of blue, red, green and yellow jelly beans (at least 15 of each). Jelly beans of the same colour are considered identical. Suppose a handful of jelly beans is taken from the bowl.

   (a) How many different handfuls of 15 jelly beans are possible?

   $n = 4, \ r = 15$

   $C(n+r-1, r) = C(4+15-1, 15) = C(18, 15) = 816$

   (b) How many different handfuls of 15 jelly beans are possible having exactly 3 red jelly beans and at least 4 green jelly beans?

   $3+4=7$ jelly beans already accounted for; the remaining $r = 15-7 = 8$ jelly beans can be any of $n=3$ colours (blue, green, yellow)

   $C(n+r-1, r) = C(3+8-1, 8) = C(10, 8) = 45$

7. (2 marks) What is the probability that a five-card poker hand from a standard deck of 52 cards contains exactly one ace? Round your answer to three decimal places.

   1 of 4 aces and 4 of the other 48 cards

   $$\frac{C(4,1) \cdot C(48,4)}{C(52,5)} = \frac{778\,320}{2\,598\,960} \approx 0.299$$

8. (1 mark) Find a recurrence relation for the number of ways, a_n, of depositing n cents into a vending machine using just nickels (5-cent coins) and dimes (10-cent coins) if the order in which the coins are deposited matters.

   $a_n = a_{n-5} + a_{n-10}$ for $n \geq 10$

9. (2 marks) Suppose f satisfies the divide-and-conquer recurrence relation

$$f(n) = 3f(n/2) + 5,$$

whenever $n = 2^k$ for $k \in \mathbb{Z}^+$. If $f(1) = 4$, then find $f(131072)$. Recall

$$f(n) = \begin{cases} f(1) + c\log_b n, & \text{if } a = 1, \\ \left[f(1) + \dfrac{c}{a-1}\right]n^{\log_b a} - \dfrac{c}{a-1}, & \text{if } a > 1. \end{cases}$$

$$f(n) = \left(4 + \frac{5}{3-1}\right)n^{\log_2 3} - \frac{5}{3-1} = \frac{13}{2}n^{\log_2 3} - \frac{5}{2}$$

$$f(131072) = \frac{13}{2}(131072)^{\log_2 3} - \frac{5}{2} = 839,411,057.$$

10. (4 marks) Consider the sequence of Fibonacci numbers $f_0, f_1, f_2, f_3, \ldots$. Recall that $f_0 = 0$ and $f_1 = 1$.

(a) State the recurrence relation that defines f_n in terms of previous Fibonacci numbers and list the Fibonacci numbers $f_2, f_3, f_4, f_5, f_6, f_7$ in the space provided.

$f_n =$ $f_{n-1} + f_{n-2}$ for $n \geq 2$. The sequence begins 0, 1, 1, 2, 3, 5, 8, 13, ...

(b) Use **strong induction** to prove that $f_n \geq n$ for integers $n \geq 5$.

Base case: $f_5 = 5 \geq 5$ and $f_6 = 8 \geq 6$ ✓

Inductive step: Suppose $k \geq 6$ and $f_j \geq j$ for all $5 \leq j \leq k$.

Then in particular $f_k \geq k$ and $f_{k-1} \geq k-1$.

Thus $f_{k+1} = f_k + f_{k-1} \geq k + (k-1) \geq k+1$

$\therefore$ By strong induction $f_n \geq n$ for all $n \geq 5$.