CAMOSUN

Name: SOLUTIONS

MATH 126 (Winter, 2015)
Term Test 3
by George Ballinger

Answer the questions in the space provided.
This test has 7 questions for a total of 25 marks.
Express answers to counting problems as integers and show your calculations.

1. (5 marks) Consider the set of 36 alphanumeric characters consisting of 26 letters A, B, C, ... Z, and 10 digits 0, 1, 2, ... 9. How many strings of six alphanumeric characters are there

   (a) if characters **cannot** be repeated?

   $P(36,6) = 36\cdot35\cdot34\cdot33\cdot32\cdot31 = 1\,402\,410\,240$

   (b) if characters **can** be repeated?

   $36^6 = 2\,176\,782\,336$

   (c) that contain exactly two A's if characters **can** be repeated?

   $C(6,2)\cdot 35^4 = 22\,509\,375$

   (d) that start with a letter or end with a digit if characters **can** be repeated?

   P.I.E.: $26\cdot36^5 + 36^5\cdot10 - 26\cdot36^4\cdot10 = 1\,740\,082\,176$

2. (3 marks) A bakery sells 18 varieties of donuts and 10 types of muffins. In how many ways can the bakery package together 12 treats, all of which are **different** from each other, consisting of

   (a) eight donuts and four muffins?

   $$C(18,8)\cdot C(10,4) = 9\ 189\ 180$$

   (b) $\underbrace{\text{at most one muffin}}_{0 \text{ or } 1}$?

   $$C(18,12)\cdot C(10,0) + C(18,11)\cdot C(10,1) = 336\ 804$$

3. (2 marks) Find the coefficient of x^8y^5 in the binomial expansion of $(3x^2 - 2y)^9$.

   $$(3x^2-2y)^9 = \sum_{j=0}^{9} \binom{9}{j} (3x^2)^{9-j}(-2y)^j = \sum_{j=0}^{9} \binom{9}{j} 3^{9-j}(-2)^j x^{18-2j} y^j$$

   for $j=5$, coeff. of x^8y^5 is $\binom{9}{5}3^4(-2)^5 = -326\ 592$

4. (1 mark) Find a recurrence relation for the number of ways, a_n, of forming n cents of postage (for $n \geq 10$) using just 3-cent and 10-cent stamps if the order in which the stamps are used to form n cents matters.

   $$a_n = a_{n-3} + a_{n-10}$$

5. (7 marks) Cupid buys a snack pack of Valentine's Day themed Smarties. Each snack pack contains a mix of 15 red, pink or white Smarties. Smarties of the same colour are considered indistinguishable.

(a) How many combinations of snack packs are possible?

$n=3,\ r=15$

$C(3+15-1,15) = C(17,15) = 136$

(b) If every snack pack is guaranteed to have at least $\underbrace{\text{5 red Smarties}}_{\text{10 left}}$, then how many combinations of snack packs are possible?

$n=3,\ r=10$

$C(3+10-1,10) = C(12,10) = 66$

Cupid opens his snack pack and discovers that it contains 7 red, 5 pink and 3 white Smarties. He then eats the Smarties one at a time by choosing them at random without paying attention to their colour.

(c) What is the probability that he eats a white Smartie first?

$\frac{3}{15} = \frac{1}{5}$ OR 0.2

(d) How many Smarties must he eat to guarantee that he eats at least three of the same colour?

7

(e) In how many different ways can he eat the Smarties?

$\frac{15!}{7!\,5!\,3!} = 360360$ OR $C(15,7)\cdot C(8,5)\cdot C(3,3) = 360360$

(f) What is the probability that he eats all the red Smarties last?

$\#\text{ways} = \frac{8!}{5!\,3!} = 56$ OR $C(8,5)\cdot C(3,3) = 56$

$\therefore \text{probability} = \frac{56}{360360} = \frac{1}{6435}$ OR 0.0001554

6. (3 marks) Suppose f satisfies the divide-and-conquer recurrence relation

$$f(n) = 4f(n/3) + 2,$$

whenever $n = 3^k$ for $k \in \mathbb{Z}^+$.

(a) If $f(9) = 10$, then find $f(81)$.

$f(27) = 4f(9) + 2 = 4(10) + 2 = 42$

$f(81) = 4f(27) + 2 = 4(42) + 2 = 170$

(b) If $f(1) = 5$, then find $f(1594323)$ by using the formula

$$f(n) = \begin{cases} f(1) + c\log_b n, & \text{if } a = 1, \\ \left[f(1) + \dfrac{c}{a-1}\right] n^{\log_b a} - \dfrac{c}{a-1}, & \text{if } a > 1. \end{cases}$$

$f(n) = \left[5 + \frac{2}{4-1}\right] n^{\log_3 4} - \frac{2}{4-1} = \frac{17}{3} n^{\log_3 4} - \frac{2}{3}$

$\therefore f(1594323) = \frac{17}{3}(1594323)^{\log_3 4} - \frac{2}{3} = \frac{17}{3}(67108864) - \frac{2}{3} = 380283562$

7. (4 marks) Define a function f recursively as follows:

$$f(0) = 2,$$
$$f(1) = 4,$$
$$f(n) = 3f(n-1) + 4f(n-2), \text{ for } n \geq 2.$$

Use strong induction to prove that $f(n) \geq 4^n$ for all $n \geq 0$.

Base case: $f(0) = 2 \geq 1 = 4^0$ and $f(1) = 4 \geq 4^1$ ✓

Inductive step: Let $k \geq 1$ and suppose $f(j) \geq 4^j$ for all $0 \leq j \leq k$.

Then in particular $f(k) \geq 4^k$ and $f(k-1) \geq 4^{k-1}$.

Thus $f(k+1) = 3f(k) + 4f(k-1)$ by recurrence relation

$\geq 3 \cdot 4^k + 4 \cdot 4^{k-1}$ by inductive hypothesis

$= 3 \cdot 4^k + 4^k$

$= 4 \cdot 4^k$

$= 4^{k+1}$

$\therefore$ By strong induction, $f(n) \geq 4^n$ for all $n \geq 0$.