CAMOSUN

Name: SOLUTIONS

MATH 126 (Winter, 2018)
Term Test 3
by George Ballinger

Answer the questions in the space provided. This test has 8 questions for a total of 30 marks. Express answers to counting problems as integers and show your calculations.

1. (4 marks) The Greek alphabet consists of 24 letters $\{\mathrm{A}, \mathrm{B}, \Gamma, \Delta, \ldots, \Omega\}$. How many strings of seven uppercase letters from the Greek alphabet are there

   (a) if letters cannot be repeated?

   $P(24,7) = 1744\,364{,}160$

   (OR $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18$)

   (b) that have exactly one Ω if letters can be repeated?

   $7 \cdot 23^6 = 1\,036\,251\,223$

2. (6 marks) How many strings of length 14 are there, using symbols from $\{M, A, T, H\}$, if

   (a) each string contains exactly two M's, three A's, four T's, and five H's?

   $\dfrac{14!}{2!\,3!\,4!\,5!} = 2\,522\,520$

   (OR $C(14,2) \cdot C(12,3) \cdot C(9,4) \cdot C(5,5)$)

   (b) each string contains at least one M?

   $4^{14} - 3^{14} = 263\,652\,487$

   (c) each string either begins with M or ends with H?

   $\underbrace{4^{13} + 4^{13} - 4^{12}}_{\text{P.I.E.}} = 117\,440\,512$

3. (2 marks) Suppose 11 Americans (including Donald Trump) and 17 Canadians (including Doug Ford) have volunteered to participate in a clinical trial studying erectile dysfunction. Suppose 14 of these 28 volunteers are selected to receive an experimental drug while the rest are to get a placebo. In how many ways can the selection of these 14 be made if there must be equal numbers of Americans and Canadians and the selection must include both Donald Trump and Doug Ford?

7 Americans (6 + Trump)

7 Canadians (6 + Ford)

$C(1,1) \cdot C(10,6) \cdot C(1,1) \cdot C(16,6) = 210 \cdot 8008$

$= 1\,681\,680$

4. (6 marks) Passengers aboard flight ABC123 can order any one of five meals: steak, chicken, fish, pasta or vegetarian. Suppose each of the passengers orders a meal.

(a) What is the minimum number of passengers required to guarantee that at least 4 ordered the same meal?

$3 \cdot 5 + 1 = 16$

(b) If there are 25 passengers, then how many different combinations of orders are possible?

$C(5+25-1, 25) = C(29,25) = 23\,751$

(c) If there are 25 passengers, then how many different combinations of orders are possible if exactly six passengers order fish and at least three order pasta?

$C(4+16-1, 16) = C(19,16) = 969$

5. (2 marks) What is the probability that a five-card poker hand from a standard deck of 52 cards contains exactly three 6's. Round your answer to six decimal places.

$$\frac{C(4,3)\cdot C(48,2)}{C(52,5)} = \frac{4\cdot 1128}{2598960} \approx 0.001736$$

6. (1 mark) Evaluate $C(30,0) + C(30,1) + C(30,2) + C(30,3) + \cdots + C(30,30)$.

$$\sum_{K=0}^{30} \binom{30}{K} = 2^{30} = 1073741824$$

7. (2 marks) The number of comparisons, $f(n)$, required (in the worst case) to sort a list of n elements using the merge sort algorithm can be described recursively by $f(1) = 0$ and

$$f(n) = 2f(n/2) + (n-1),$$

whenever $n = 2^k$ for $k \in \mathbb{Z}^+$. How many comparisons are required to sort a list of size $n = 16$?

$$f(2) = 2f(1) + 1 = 2(0)+1 = 1$$
$$f(4) = 2f(2)+3 = 2(1)+3 = 5$$
$$f(8) = 2f(4)+7 = 2(5)+7 = 17$$
$$f(16) = 2f(8)+15 = 2(17)+15 = 49$$

$\therefore$ 49 comparisons are needed.

8. (7 marks)

(a) Use **strong induction** to prove that every amount of postage of 12 cents or more can be formed using just 3-cent and 7-cent stamps.

Base case: 12¢ can be formed using 4 3¢ stamps
13¢ can be formed using 2 3¢ and 1 7¢ stamps
14¢ can be formed using 2 7¢ stamps

Assume $k \geq 14$ and j¢ of postage can be formed for all $12 \leq j \leq k$. Since $k \geq 14$ then $k-2 \geq 12$ and so in particular $(k-2)$¢ can be formed. Adding a 3¢ stamp to $(k-2)$¢ gives $(k+1)$¢ and so $(k+1)$¢ can be formed. So by strong induction, n¢ can be formed using 3¢ and 7¢ stamps for all $n \geq 12$. ∎

(b) Let a_n represent the number of ways of forming n cents of postage using just 3-cent and 7-cent stamps if the order in which the stamps are used to form n cents matters.

(i) Find a_{13}.

3¢, 3¢, 7¢
3¢, 7¢, 3¢
7¢, 3¢, 3¢

3 ways $\therefore a_{13} = 3$

(ii) Find a recurrence relation for a_n.

$$a_n = a_{n-3} + a_{n-7} \text{ for } n \geq 7$$

(iii) Find a_{50} given the following values for a_n.

n	12	13	14	15	16	$\cdots$	42	43	44	45	46	47	48	49	50
a_n	1	?	1	1	4	$\cdots$	122	139	122	173	224	204	249	346	?

$$a_{50} = a_{47} + a_{43} = 204 + 139 = 343$$