CAMOSUN

Name: SOLUTIONS

MATH 126 (Winter, 2014)
Term Test 2
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Answer the questions in the space provided.
This test has 12 questions for a total of 25 marks.

1. (2 marks) Consider the factorial function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n!$ where $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers. Is this function one-to-one? Is this function onto? Briefly justify your answers.

$f(0) = 1 = f(1)$ $\therefore$ f is not one-to-one

$f(n) \neq 0$ for any $n \in \mathbb{N}$ $\therefore$ f is not onto

2. (1 mark) If $f(n) = \lceil n/2 \rceil$ and $S = \{0, 1, 2, 3, 4\}$, then find $|f(S)|$, i.e. the cardinality of the image of S. Recall that $\lceil x \rceil$ denotes the ceiling of x.

$f(S) = \{0, 1, 2\}$ $\therefore$ $|f(S)| = 3$

3. (3 marks) Prove that $f(x) = \log_2(x^3 + 1)$ is $O(\log_2 x)$ by finding positive constants (i.e. witnesses) C and k from the definition of big-O.

$$
\begin{aligned}
|f(x)| &= \log_2(x^3+1) && \text{for } x>0 \\
&\leq \log_2(x^3+x^3) && \text{for } x>1 \\
&= \log_2 2x^3 \\
&\leq \log_2 x^4 && \text{for } x>2 \\
&= 4\log_2 x
\end{aligned}
$$

$\therefore$ f is $O(\log_2 x)$ with $C=4$ and $k=2$ as witnesses. ■

4. (3 marks) Express the greatest common divisor of 14453 and 7081 as a linear combination of 14453 and 7081.

$$14453 = 7081 \cdot 2 + 291$$
$$7081 = 291 \cdot 24 + 97 \leftarrow \text{gcd}$$
$$291 = 97 \cdot 3 + 0$$

$$97 = 7081 - 291 \cdot 24 = 7081 - (14453 - 7081 \cdot 2) \cdot 24$$
$$= 7081 - 24 \cdot 14453 + 48 \cdot 7081$$
$$= 49 \cdot 7081 - 24 \cdot 14453$$

5. (1 mark) List all the positive integers less than 10 that are relatively prime to 10.

1, 3, 7, 9

6. (1 mark) Find two integers, one positive and the other negative, that are congruent to 15 modulo 7.

1, 8, 22, 29, 36, etc.

−6, −13, −20, −27, etc.

7. (1 mark) Convert the hexadecimal number $(2000AD)_{16}$ to base 10 without using your calculator's number system conversion keys (i.e. show your calculations).

$$(2000AD)_{16} = 2 \times 16^5 + 10 \times 16^1 + 13 \times 16^0$$
$$= 2097152 + 160 + 13$$
$$= 2097325$$

8. (2 marks) Suppose a sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = na_{n-1} + 3a_{n-2} - 6$$

and the initial conditions $a_0 = 5$ and $a_1 = 2$. Find a_2, a_3 and a_4.

$a_2 = 2a_1 + 3a_0 - 6 = 2(2) + 3(5) - 6 = 13$

$a_3 = 3a_2 + 3a_1 - 6 = 3(13) + 3(2) - 6 = 39$

$a_4 = 4a_3 + 3a_2 - 6 = 4(39) + 3(13) - 6 = 189$

9. (2 marks) Prove that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable by showing that the elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ can be listed in a sequence $\{a_n\} = a_1, a_2, a_3, \ldots$. List at least the first 10 terms of your sequence so as to clearly show how your sequence is constructed. Note that elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ may be tabulated as follows:

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

$(1,1), (2,1), (1,2), (1,3), (2,2), (3,1), (4,1), (3,2), (2,3), (1,4), (1,5), \ldots$

List ordered pairs (a,b) where $a+b=2$, then $a+b=3$, then $a+b=4$, etc. following a path like the one shown above.

10. (3 marks) Prove that if $f: B \to C$ is one-to-one and $g: A \to B$ is one-to-one, then $f \circ g$ is one-to-one.

Suppose f and g are one-to-one. Let $x_1, x_2 \in A$. Suppose $(f \circ g)(x_1) = (f \circ g)(x_2)$. Then $f(g(x_1)) = f(g(x_2))$. Since f is one-to-one then $g(x_1) = g(x_2)$ and since g is one-to-one then $x_1 = x_2$. Thus $f \circ g$ is one-to-one. ■

11. (3 marks) Use the definitions of *congruence* and *divides* to prove that for all $x, y, m, n \in \mathbb{Z}$ with $m, n > 0$, if $n \mid m$ and $x \equiv y \pmod{m}$, then $x \equiv y \pmod{n}$.

Suppose $n \mid m$ and $x \equiv y \pmod{m}$. Then $n \mid m$ and $m \mid (x-y)$ by def'n of congruence and so $m = nk$ and $x - y = mj$ for some $k, j \in \mathbb{Z}$ by def'n of divides. Thus $x - y = (nk)j = n(kj) = ni$ where $i = kj \in \mathbb{Z}$, proving $n \mid (x-y)$ and so $x \equiv y \pmod{n}$. ■

12. (3 marks) Use the Principle of Mathematical Induction to prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}$$

for every positive integer n.

<u>Proof</u> Base case $n=1$ $\quad \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{(1+1)}$ ✓

Inductive step: Suppose $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$ where $k \in \mathbb{Z}^+$. Then

$$\underbrace{\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)}} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{by ind. hyp.}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

$\therefore$ By PMI statement is true for all $n \in \mathbb{Z}^+$. ■