CAMOSUN

Name: ____________________

MATH 126 (Winter, 2014)
Term Test 2

by George Ballinger

Answer the questions in the space provided. This test has 12 questions for a total of 25 marks.

1. (2 marks) Consider the factorial function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n!$ where $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers. Is this function one-to-one? Is this function onto? Briefly justify your answers.

2. (1 mark) If $f(n) = \lceil n/2 \rceil$ and $S = \{0, 1, 2, 3, 4\}$, then find $|f(S)|$, i.e. the cardinality of the image of S. Recall that $\lceil x \rceil$ denotes the ceiling of x.

3. (3 marks) Prove that $f(x) = \log_2(x^3 + 1)$ is $O(\log_2 x)$ by finding positive constants (i.e. witnesses) C and k from the definition of big-O.

4. (3 marks) Express the greatest common divisor of 14453 and 7081 as a linear combination of 14453 and 7081.

5. (1 mark) List all the positive integers less than 10 that are relatively prime to 10.

6. (1 mark) Find two integers, one positive and the other negative, that are congruent to 15 modulo 7.

7. (1 mark) Convert the hexadecimal number $(2000AD)_{16}$ to base 10 without using your calculator's number system conversion keys (i.e. show your calculations).

8. (2 marks) Suppose a sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = na_{n-1} + 3a_{n-2} - 6$$

and the initial conditions $a_0 = 5$ and $a_1 = 2$. Find a_2, a_3 and a_4.

9. (2 marks) Prove that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable by showing that the elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ can be listed in a sequence $\{a_n\} = a_1, a_2, a_3, \ldots$. List at least the first 10 terms of your sequence so as to clearly show how your sequence is constructed. Note that elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ may be tabulated as follows:

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

10. (3 marks) Prove that if $f : B \to C$ is one-to-one and $g : A \to B$ is one-to-one, then $f \circ g$ is one-to-one.

11. (3 marks) Use the definitions of *congruence* and *divides* to prove that for all $x, y, m, n \in \mathbb{Z}$ with $m, n > 0$, if $n \mid m$ and $x \equiv y (\text{mod } m)$, then $x \equiv y (\text{mod } n)$.

12. (3 marks) Use the Principle of Mathematical Induction to prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}$$

for every positive integer n.