CAMOSUN

Name: SOLUTIONS

MATH 126 (Winter, 2015)
Term Test 2
by George Ballinger

Answer the questions in the space provided. This test has 12 questions for a total of 25 marks.

1. (2 marks) Consider the function $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = \lceil (n+1)/2 \rceil$ and $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers. Recall that $\lceil x \rceil$ denotes the ceiling of x.

   (a) Is f one-to-one? Briefly justify your answer.

   No. $f(0) = 1 = f(1)$

   (b) Is f onto? Briefly justify your answer.

   No. $f(n) \neq 0$ for any $n \in \mathbb{N}$

2. (3 marks)

   (a) Define $g : \mathbb{R} \to \mathbb{R}$ to be the absolute value function $g(x) = |x|$. Let $X = \{-2, -1, 0, 1\}$ and $Y = \{0, 1, 2, 3\}$. Evaluate both $g(X \cap Y)$ and $g(X) \cap g(Y)$.

   $g(X \cap Y) = g(\{0,1\}) = \{0,1\}$

   $g(X) \cap g(Y) = \{0,1,2\} \cap \{0,1,2,3\} = \{0,1,2\}$

   (b) In general, what property must a function $f : A \to B$ have in order to guarantee that $f(S \cap T) = f(S) \cap f(T)$ for all $S \subseteq A$ and $T \subseteq A$?

   f must be one-to-one

3. (1 mark) Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(a) = a \,\mathbf{mod}\, 5$. What is the range of f?

   $\{0,1,2,3,4\}$ all possible remainders upon division by 5

4. (2 marks) Show that the sequence $\{a_n\}$, where $a_n = 3^n$, is a solution of the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$.

$$a_{n-1} + 6a_{n-2} = 3^{n-1} + 6 \cdot 3^{n-2} = 3^{n-1} + 2 \cdot 3 \cdot 3^{n-2}$$

$$= 3^{n-1} + 2 \cdot 3^{n-1} = 3 \cdot 3^{n-1} = 3^n = a_n \quad .$$

5. (1 mark) Show that the set $S = \{n\pi | \, n \in \mathbb{Z}\}$ of integer multiples of π is countable. Listing the elements of S in a sequence will suffice, provided it's clear how the sequence is constructed.

$$S = \{0, \pi, -\pi, 2\pi, -2\pi, 3\pi, -3\pi, \ldots\}$$

6. (3 marks) Prove that $f(x)$ is $O(x^2)$, where

$$f(x) = \frac{6x^3 + x^2 + 7}{\sqrt{4x^2 + 9}},$$

by finding positive constants ("witnesses") C and k from the definition of big-O.

$$|f(x)| = \frac{6x^3 + x^2 + 7}{\sqrt{4x^2+9}} \quad \text{for } x > 0$$

$$\leq \frac{6x^3 + x^3 + 7x^3}{\sqrt{4x^2+9}} \quad \text{for } x > 1$$

$$= \frac{14x^3}{\sqrt{4x^2+9}} \quad \text{for } x > 1$$

$$\leq \frac{14x^3}{\sqrt{4x^2}} \quad \text{for } x > 1$$

$$= \frac{14x^3}{2x} \quad \text{for } x > 1$$

$$= 7x^2 \quad \text{for } x > 1$$

$\therefore$ f is $O(x^2)$ with $C = 7$ and $k = 1$ as witnesses.

7. (3 marks)

(a) Use the Euclidean Algorithm to find the greatest common divisor of 4891 and 12191.

$12191 = 4891 \cdot 2 + 2409$

$4891 = 2409 \cdot 2 + 73 \leftarrow \gcd = 73$

$2409 = 73 \cdot 33 + 0$

(b) Find the least common multiple of 4891 and 12191.

$$\text{lcm} = \frac{4891 \cdot 12191}{73} = 816{,}797$$

8. (1 mark) List all the positive integers less than 12 that are relatively prime to 12.

1, 5, 7, 11

9. (1 mark) If baby Jane is born on a Friday, then on what day of the week will she have her first birthday, 365 days later?

$365 = 7 \cdot 52 + 1$

$365 \bmod 7 = 1$ $\therefore$ 1 day after Friday, ie Saturday

10. (1 mark) Convert the hexadecimal number $(\text{C0FFEE})_{16}$ to base 10 without using your calculator's number system conversion keys (i.e. show your calculations).

$$(\text{C0FFEE})_{16} = 12 \cdot 16^5 + 0 \cdot 16^4 + 15 \cdot 16^3 + 15 \cdot 16^2 + 14 \cdot 16^1 + 14 \cdot 16^0$$

$$= 12{,}582{,}912 + 61{,}440 + 3{,}840 + 224 + 14$$

$$= 12{,}648{,}430$$

11. (4 marks)

(a) Find a positive integer m such that $6^2 \equiv 4^2 \pmod{m}$ but $6 \not\equiv 4 \pmod{m}$.

$$m \mid \underbrace{(6^2-4^2)}_{20} \qquad m \nmid \underbrace{(6-4)}_{2}$$

m must divide 20, but not 2. $\therefore$ m can be 4, 5, 10 or 20

(b) Use the definitions of *congruence* and *divides* to prove that for all $a, b, m \in \mathbb{Z}$ with $m > 0$, if $a \equiv b \pmod{m}$, then $a^2 \equiv b^2 \pmod{m}$. You may not use any theorems about congruence or divisibility (only their definitions) unless you also prove them.

Suppose $a \equiv b \pmod{m}$. Then $m \mid (a-b)$ and so $a-b = mk$ for some $k \in \mathbb{Z}$. Thus

$a^2 - b^2 = (a+b)(a-b) = (a+b)mk = jm$ where $j = (a+b)k \in \mathbb{Z}$.

$\therefore m \mid (a^2-b^2)$ and so $a^2 \equiv b^2 \pmod{m}$. ∎

12. (3 marks) Use the Principle of Mathematical Induction to prove that $1+3+5+\cdots+(2n-1) = n^2$ for every positive integer n.

Base case: $1 = 1^2$ ✓

Ind. step: Suppose $1+3+5+\cdots+(2k-1) = k^2$ where $k \in \mathbb{Z}^+$.

Then $1+3+5+\cdots+(2k-1)+(2(k+1)-1)$

$= \underbrace{}_{k^2} + (2k+1)$ by ind. hyp.

$= k^2 + 2k + 1$

$= (k+1)^2$

$\therefore$ By PMI, $1+3+5+\cdots+(2n-1) = n^2$ for all $n \in \mathbb{Z}^+$. ∎