CAMOSUN

Name: ______________________

MATH 126 (Winter, 2015)
Term Test 2
by George Ballinger

Answer the questions in the space provided. This test has 12 questions for a total of 25 marks.

1. (2 marks) Consider the function $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = \lceil (n+1)/2 \rceil$ and $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers. Recall that $\lceil x \rceil$ denotes the ceiling of x.

   (a) Is f one-to-one? Briefly justify your answer.

   (b) Is f onto? Briefly justify your answer.

2. (3 marks)

   (a) Define $g : \mathbb{R} \to \mathbb{R}$ to be the absolute value function $g(x) = |x|$. Let $X = \{-2, -1, 0, 1\}$ and $Y = \{0, 1, 2, 3\}$. Evaluate both $g(X \cap Y)$ and $g(X) \cap g(Y)$.

   (b) In general, what property must a function $f : A \to B$ have in order to guarantee that $f(S \cap T) = f(S) \cap f(T)$ for all $S \subseteq A$ and $T \subseteq A$?

3. (1 mark) Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(a) = a \,\mathbf{mod}\, 5$. What is the range of f?

4. (2 marks) Show that the sequence $\{a_n\}$, where $a_n = 3^n$, is a solution of the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$.

5. (1 mark) Show that the set $S = \{n\pi |\, n \in \mathbb{Z}\}$ of integer multiples of π is countable. Listing the elements of S in a sequence will suffice, provided it's clear how the sequence is constructed.

6. (3 marks) Prove that $f(x)$ is $O(x^2)$, where

$$f(x) = \frac{6x^3 + x^2 + 7}{\sqrt{4x^2 + 9}},$$

by finding positive constants ("witnesses") C and k from the definition of big-O.

7. (3 marks)

   (a) Use the Euclidean Algorithm to find the greatest common divisor of 4891 and 12191.

   (b) Find the least common multiple of 4891 and 12191.

8. (1 mark) List all the positive integers less than 12 that are relatively prime to 12.

9. (1 mark) If baby Jane is born on a Friday, then on what day of the week will she have her first birthday, 365 days later?

10. (1 mark) Convert the hexadecimal number $(\mathrm{C0FFEE})_{16}$ to base 10 without using your calculator's number system conversion keys (i.e. show your calculations).

11. (4 marks)

(a) Find a positive integer m such that $6^2 \equiv 4^2 \pmod{m}$ but $6 \not\equiv 4 \pmod{m}$.

(b) Use the definitions of *congruence* and *divides* to prove that for all $a, b, m \in \mathbb{Z}$ with $m > 0$, if $a \equiv b \pmod{m}$, then $a^2 \equiv b^2 \pmod{m}$. You may not use any theorems about congruence or divisibility (only their definitions) unless you also prove them.

12. (3 marks) Use the Principle of Mathematical Induction to prove that $1+3+5+\cdots+(2n-1) = n^2$ for every positive integer n.