CAMOSUN

Name: SOLUTIONS

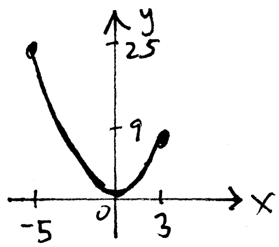
MATH 126 (Winter, 2018)

Term Test 2

by George Ballinger

Answer the questions in the space provided. This test has 11 questions for a total of 30 marks.

1. (1 mark) Given the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, find $f(A)$, where $A = [-5, 3]$ is a closed interval.



$f(A) = [0, 25]$

2. (1 mark) Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ that is one-to-one and onto but is not the identity function. You do not have to prove that your function is one-to-one and onto.

$f(x) = -x$, $f(x) = x + 1$ ↑ or some other integer, $f(x) = \begin{cases} x+1, & \text{if } x \text{ is even} \\ x-1, & \text{if } x \text{ is odd} \end{cases}$

Answers vary. etc.

3. (1 mark) Which *one* of the following expressions defines an actual function from $\mathbb{R}$ to $\mathbb{R}$? Circle your answer. No justification is required.

multi-valued ∴ not a function

Domains of (i), (ii), (iv) and (vi) are not all of $\mathbb{R}$.

(i) $f(x) = \sqrt{x}$	(iii) $f(x) = \pm x$	(v) $f(x) = \lfloor x \rfloor$
(ii) $f(x) = 1/x$	(iv) $f(x) = x!$	(vi) $f(x) = \log_2 x$

4. (1 mark) Which *one* of the following sequences $\{a_n\}$ is *not* a solution of the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$. Circle your answer. No justification is required.

(i) $\{0\} = 0, 0, 0, 0, 0, \ldots$ ✓	(iii) $\{(-1)^n\} = -1, 1, -1, 1, -1, \ldots$ ✓
(ii) $\{n-1\} = 0, 1, 2, 3, 4, \ldots$	(iv) $\{3^n\} = 3, 9, 27, 81, 243, \ldots$ ✓

5. (2 marks) If the time now is 2pm, then what time of day was it 10,000 hours ago?

$-10000 = 24 \cdot (-417) + 8$

$-10000 \text{ mod } 24 = 8$

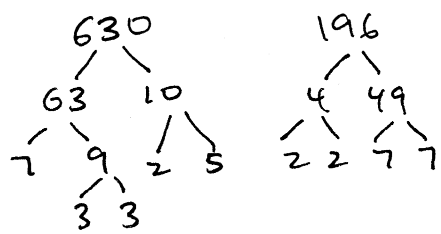
8 hr after 2pm is <u>10PM</u>.

6. (4 marks) Using the definition of onto, prove that if $f : B \to C$ is onto and $g : A \to B$ is onto, then $f \circ g : A \to C$ is onto.

Assume $f: B \to C$ and $g: A \to B$ are both onto. Let $c \in C$. Since f is onto $\exists b \in B$ such that $f(b) = c$. Since g is onto $\exists a \in A$ such that $g(a) = b$. Since $a \in A$ and $(f \circ g)(a) = f(g(a)) = f(b) = c$, then $f \circ g$ is onto. ∎

7. Find the greatest common divisor of 630 and 196 by using

(a) (2 marks) the prime factorization of each number.



$630 = 2 \cdot 3^2 \cdot 5 \cdot 7$

$196 = 2^2 \cdot 7^2$

$\therefore \gcd(630, 196) = 2 \cdot 7 = 14$

(b) (2 marks) the Euclidean Algorithm.

$630 = 196 \cdot 3 + 42$

$196 = 42 \cdot 4 + 28$

$42 = 28 \cdot 1 + 14 \leftarrow \gcd$

$28 = 14 \cdot 2 + 0$

8. (2 marks) Give an example of a *composite* number n such that the numbers 14, 15 and n are pairwise relatively prime.

$14 = 2 \cdot 7$

$15 = 3 \cdot 5$

n must be product of 2 or more primes excluding 2, 3, 5, 7.

$\therefore$ possible values for n are

$n = 11^2 = 121$, $n = 11 \cdot 13 = 143$, $n = 13^2 = 169$

etc.

9. (2 marks) Prove that the union of two countably infinite sets

$$A = \{a_1, a_2, a_3, a_4, \ldots\} \text{ and } B = \{b_1, b_2, b_3, b_4, \ldots\}$$

is countably infinite. Listing the elements of $A \cup B$ in a sequence will suffice, provided it's clear how the sequence is constructed.

$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \ldots\}$

$\therefore$ $A \cup B$ is Countably infinite.

10. Consider four-digit octal (base-8) numbers of the form $n = (abcd)_8$.

(a) (2 marks) Convert $(7143)_8$ and $(1060)_8$ to decimal (base-10), showing your calculations.

$(7143)_8 = 7 \cdot 8^3 + 1 \cdot 8^2 + 4 \cdot 8^1 + 3 \cdot 8^0 = 3584 + 64 + 32 + 3 = (3683)_{10}$

$(1060)_8 = 1 \cdot 8^3 + 6 \cdot 8^1 = 512 + 48 = (560)_{10}$

(b) (3 marks) Prove that if $n = (abcd)_8$, then $n \equiv d \pmod 8$. You may not use any theorems about congruence or divisibility (only their definitions) unless you also prove them.

If $n = (abcd)_8$, then

$n = a \cdot 8^3 + b \cdot 8^2 + c \cdot 8^1 + d \cdot 8^0 = 8(\underbrace{a \cdot 8^2 + b \cdot 8 + c}_{K}) + d$

$\therefore$ $n - d = 8K$ where $K = a \cdot 8^2 + b \cdot 8 + c \in \mathbb{Z}$.

Thus $8 \mid (n-d)$ and so $n \equiv d \pmod 8$. ■

(c) (1 mark) Convert $(7143)_8$ to binary (base-2). You do not need to show any work.

$(7143)_8 = (\underbrace{111}_{7}\ \underbrace{001}_{1}\ \underbrace{100}_{4}\ \underbrace{011}_{3})_2$

11.

(a) (4 marks) Use the Principle of Mathematical Induction to prove that $n! \geq 3(2^n)$ for all integers $n \geq 5$.

Base case $n=5$: $5! = 120$ and $3(2^5) = 96$
and $120 \geq 96$ ✓

Inductive step: Suppose $k! \geq 3(2^k)$ where $k \geq 5$:

Then $(k+1)! = (k+1)k! \geq (k+1)(3)(2^k)$ by ind. hyp

$\geq (1+1)(3)(2^k)$

$= 2 \cdot (3)(2^k)$

$= 3(2^{k+1})$.

$\therefore$ By PMI, $n! \geq 3(2^n)$ for all $n \geq 5$.

(b) (2 marks) Prove that 2^n is $O(n!)$ by finding positive constants (i.e. witnesses) C and k from the definition of big-O.

From part (a) $n! \geq 3(2^n)$ for $n \geq 5$

$\therefore$ $2^n \leq \frac{1}{3} n!$ for $n \geq 5$

C k

So $n!$ is $O(2^n)$ with $C = 1/3$ and $k=5$
as witnesses