CAMOSUN

Name: SOLUTIONS

# MATH 126 (Winter, 2014)
## Term Test 1
### by George Ballinger

Answer the questions in the space provided.
This test has 16 questions for a total of 25 marks.

1. (2 marks) Construct a truth table for the proposition $(p \to q) \land (p \lor \neg q)$.

| $p$ | $q$ | $p \to q$ | $\neg q$ | $p \lor \neg q$ | $(p \to q) \land (p \lor \neg q)$ |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | F | T | T | F |
| F | T | T | F | F | F |
| F | F | T | T | T | T |

2. (1 mark) Which of the following propositions is a contradiction? Circle your answer. No justification is required.

(i) $p \land p$ (ii) $p \lor p$ (iii) $p \oplus p$ (iv) $p \to p$ (v) $p \leftrightarrow p$ (vi) $\neg p$

3. (1 mark) Find a compound proposition involving the propositional variables $p$, $q$ and $r$ that is true when $p$, $q$ and $r$ are all false but is false otherwise.

$\neg p \land \neg q \land \neg r$ or $\neg(p \lor q \lor r)$

4. (1 mark) Write the statement, "I will graduate only if I pass this course," in the form, "If ..., then ...."

If I graduate, then I've passed the course.

5. (1 mark) Write the **converse** of the statement, "For a function to be differentiable it is necessary that it be continuous."

For a function to be differentiable it is sufficient that it be continuous.

or If a function is continuous, then it is differentiable.

6. (1 mark) Write the negation of the statement, "If I travel overseas, then I get sick." (Do not simply use words like "It is not the case that ....")

I travel overseas and I do not get sick.

7. (2 marks) Using a proof by contradiction, show that there does not exist a smallest positive rational number.

Suppose there does exist a smallest positive rational number $x$ of the form $x=\frac{p}{q}$ where $p$ and $q$ are integers and $q \neq 0$. Consider $\frac{1}{2}x = \frac{p}{2q}$ (or consider $\frac{p}{q+1}$). It too is a positive rational number, yet it is clearly smaller than $x$. This contradicts the claim that $x$ is smallest, which in turn proves the statement. ■

8. (3 marks) Let $m$ and $n$ be integers. Use a proof by contraposition to prove the statement, "If $mn$ is even, then either $m$ is even or $n$ is even."

Suppose it is not the case that $m$ is even or $n$ is even. Then both $m$ and $n$ are odd. Thus $m = 2i+1$ and $n = 2j+1$ for some integers $i$ and $j$. Therefore,

$$\begin{aligned} mn = (2i+1)(2j+1) &= 4ij + 2i + 2j + 1 \\ &= 2(2ij+i+j)+1 \\ &= 2k+1 \quad \text{where } k = 2ij+i+j \in \mathbb{Z}. \end{aligned}$$

implying $mn$ is odd, and thus not even.

This proves (by contraposition) that if $mn$ is even then $m$ or $n$ is even. ■

9. (2 marks) Using only the logical equivalences listed on the "Logical Equivalences" supplement, prove $\neg(p \wedge \neg q) \vee q \equiv p \rightarrow q$. At each step identify the logical equivalence being used.

$$
\begin{aligned}
\neg(p \wedge \neg q) \vee q &\equiv (\neg p \vee \neg(\neg q)) \vee q && \text{De morgan} \\
&\equiv (\neg p \vee q) \vee q && \text{double negation} \\
&\equiv \neg p \vee (q \vee q) && \text{associativity} \\
&\equiv \neg p \vee q && \text{idempotent} \\
&\equiv p \rightarrow q && \text{equivalence of implication}
\end{aligned}
$$

10. (2 marks) Define $P(x, y)$ to be, "student $x$ got question $y$ correct," where the domain for $x$ is the set of students and the domain for $y$ is the set of questions on a test.

(a) Write the following sentence symbolically using quantifiers: "Every student got at least one question correct."

$$\forall x \exists y \, P(x,y)$$

(b) Write the negation of the sentence from part (a) symbolically so that no negation symbols appear outside (i.e. in front of) a quantifier and then translate this negation back into English. (Do not simply use words like "It is not the case that ....")

$$\neg \forall x \exists y \, P(x,y) \equiv \exists x \forall y \, \neg P(x,y)$$

There is a student who got all the questions incorrect.

11. (1 mark) Let $P(x)$ be a propositional function. Recall that $\exists! x P(x)$ means "there exists a unique $x$ such that $P(x)$." Rewrite $\exists! x P(x)$ by using only existential and universal quantifiers instead of the uniqueness quantifier $\exists!$.

$$\exists! x P(x) \equiv \exists x (P(x) \wedge \forall y (P(y) \rightarrow (x = y)))$$

$$\text{OR} \quad \exists x \forall y \, (P(x) \wedge \underbrace{(P(y) \rightarrow (x = y))}_{\text{OR } (x \neq y) \rightarrow \neg P(y)})$$

12. (3 marks) Consider the following argument.

If I don't drive to work or I'm not in a hurry, then I walk to work.
I don't pay for parking and I have a bus pass.
If I drive to work, then I pay for parking.
$\therefore$ I walk to work.

(a) Define $p$, $q$, $r$, $s$ and $t$ as follows and rewrite the argument symbolically using logic notation.

$p$: I drive to work.
$q$: I am in a hurry.
$r$: I walk to work.
$s$: I pay for parking.
$t$: I have a bus pass.

$$(\neg p \vee \neg q) \to r$$
$$\neg s \wedge t$$
$$p \to s$$
$$\therefore r$$

(b) Prove that the argument in part (a) is valid. State the reason for each step of your proof by identifying hypotheses or citing an appropriate rule of inference or logical equivalence.

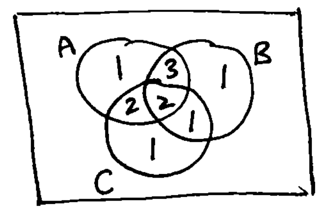
| STEP | REASON |
|---|---|
| 1. $\neg s \wedge t$ | hyp. |
| 2. $\neg s$ | Simplification (1) |
| 3. $p \to s$ | hyp. |
| 4. $\neg p$ | modus tollens (2,3) |
| 5. $\neg p \vee \neg q$ | addition (4) |
| 6. $(\neg p \vee \neg q) \to r$ | hyp. |
| 7. $r$ | modus ponens (5,6) |

13. (1 mark) Suppose $A$ is a set and $\mathcal{P}(A)$ is its power set. Which one of the following is generally not true, where $\varnothing$ denotes the empty set? Circle your answer. No justification is required.

(i) $\varnothing \in \mathcal{P}(A)$ (ii) $\varnothing \subseteq \mathcal{P}(A)$ (iii) $\varnothing \subset \mathcal{P}(A)$ (iv) $A \in \mathcal{P}(A)$ (v) $A \subseteq \mathcal{P}(A)$ (vi) $\mathcal{P}(A) \subseteq \mathcal{P}(A)$

14. (1 mark) Suppose $A$, $B$ and $C$ are finite sets satisfying the following: $|A| = 8$, $|B| = 7$, $|C| = 6$, $|A \cap B| = 5$, $|A \cap C| = 4$, $|B \cap C| = 3$ and $|A \cap B \cap C| = 2$. Find $|A \cup B \cup C|$.

$$|A\cup B\cup C| = |A|+|B|+|C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$
$$= 8+7+6-5-4-3+2 = 11$$




15. (1 mark) For each positive integer $i$, define the open interval $A_i$ by

$$A_i = (0, 1/i) = \{x \in \mathbb{R} \mid 0 < x < 1/i\}.$$

In other words, $A_1 = (0,1)$, $A_2 = (0,1/2)$, $A_3 = (0,1/3)$, etc. Find

$$\bigcup_{i=1}^{\infty} A_i = \underline{(0,1)} \qquad \bigcap_{i=1}^{\infty} A_i = \underline{\varnothing}.$$

16. (2 marks) Let $A$ and $B$ be sets. Prove that if $A \cap B = A$, then $A \subseteq B$.

Suppose $A\cap B = A$. Let $x \in A$. Then $x \in A\cap B$ since $A = A\cap B$. Thus $x \in A$ and $x \in B$ by def'n of $\cap$. Since $x \in B$ then this proves $A \subseteq B$. ∎