Methods of Proof

A **theorem** is a statement s that can be shown to be true. A **proof** is a valid argument that establishes the truth of a theorem. The statement s usually has the form $p \to q$, where p is the hypothesis (it can consist of several premises, $p \equiv p_1 \wedge p_2 \wedge \cdots \wedge p_n$) and q is the conclusion.

There are three main methods of proving a statement s of the form $p \to q$.

1. **Direct proof:** $p \to q$

   In a direct proof one assumes p is true and then deduces, through a logical sequence of steps, that q is true.

2. **Proof by Contraposition:** $\neg q \to \neg p$

   A proof by contraposition relies on the fact that an implication $p \to q$ is logically equivalent to its contrapositive $\neg q \to \neg p$. Therefore to prove a statement s of the form $p \to q$ by contraposition involves assuming q is false and then deducing (directly) that p is false.

3. **Proof by Contradiction:** $(p \wedge \neg q) \to \mathbf{F}$

   To prove a statement s by contradiction is to assume s is false and arrive at a contradiction $\mathbf{F}$. Note that $s \equiv \neg s \to \mathbf{F}$. The contradiction $\mathbf{F}$ usually has the form $\mathbf{F} \equiv r \wedge \neg r$ for some statement r; in other words some statement r is shown to be both true and false.

   If as is usually the case, the statement s to be proven is of the form $p \to q$, then since $\neg(p \to q) \equiv p \wedge \neg q$, a proof by contradiction involves proving $(p \wedge \neg q) \to \mathbf{F}$.

   So to prove $p \to q$ by contradiction one assumes p is true (as in a direct proof) and q is false (as in a proof by contraposition) and arrives at a contradiction.

If in a statement s of the form $p \to q$, p can be shown to be false, then the statement is automatically true (regardless of the truth value of q). This is called a **vacuous proof**.

Similarly, if in a statement s of the form $p \to q$, q can be shown to be true independently of p, then the statement is automatically true. This is called a **trivial proof**.