Product and Sum Rules of Counting

Product Rule: Suppose a procedure can be broken down into a sequence of two tasks. If there are n_1 ways of doing the first task and for each of these ways there are n_2 ways of doing the second task, then there are $n_1 \cdot n_2$ ways of doing the procedure.

Generalization of Product Rule: If a procedure involves performing a sequence of tasks $T_1, T_2, \ldots, T_m$, where each task T_i can be done in n_i ways regardless of how previous tasks were done, then there are $n_1 \cdot n_2 \cdot \cdots \cdot n_m$ ways of doing the procedure.

Product Rule Set Analogy: Let $A_1, A_2, \ldots, A_m$ represent the sets of $n_1, n_2, \cdots, n_m$ ways of doing tasks $T_1, T_2, \ldots, T_m$, respectively. Then the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ represents the set of ways of doing the entire procedure. Moreover,

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

Sum Rule: If a single task can be done in any of n_1 ways or n_2 ways, where none of the n_1 ways are the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Generalization of Sum Rule: If a single task can be done in any of n_1 ways or n_2 ways or ... or n_m ways, with none of these various ways being the same, then there are $n_1 + n_2 + \cdots + n_m$ ways to do the task.

Sum Rule Set Analogy: Let $A_1, A_2, \ldots, A_m$ represent the sets of $n_1, n_2, \cdots, n_m$ ways, respectively, of doing a single task. Then $A_1 \cup A_2 \cup \cdots \cup A_m$ represents the set of all ways of doing the task. Moreover, if these sets are pairwise disjoint, then

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|.$$