Big-O Example

We show that $\log n!$ is $O(n \log n)$ and $n \log n$ is $O(\log n!)$.

We use the following well-known properties of logarithms.

$$\begin{aligned} \log(ab) = \log a + \log b, &\quad \text{the log of a product equals the sum of the logs} \\ a \leq b \rightarrow \log a \leq \log b, &\quad \text{the log function is increasing} \end{aligned}$$

We first show $\log n!$ is $O(n \log n)$.

$$\begin{aligned} \log n! &= \log(1 \cdot 2 \cdot 3 \cdot \cdots \cdot n) \\ &= \log 1 + \log 2 + \log 3 + \cdots + \log n \\ &\leq \log n + \log n + \log n + \cdots + \log n \\ &= n \log n \end{aligned}$$

Therefore $\log n!$ is $O(n \log n)$ with $C = 1$ and $k = 1$ as witnesses.

Showing $n \log n$ is $O(\log n!)$ is trickier and before doing this we first prove an interesting set of inequalities. We claim that n is less than or equal to each of the following products: $1 \cdot n, 2 \cdot (n-1), 3 \cdot (n-2), 4 \cdot (n-3), \ldots, (n-1) \cdot 2, n \cdot 1$. If true, then $\log n$ will be less than or equal to the log of each of these products.

If $1 \leq i \leq n$, then $i-1 \geq 0$ and $n-i \geq 0$ and so $0 \leq (i-1)(n-i)$. Expanding this gives $0 \leq in - i^2 - n + i$. Adding n to both sides gives $n \leq in - i^2 + i$ and finally by factoring we get $n \leq i(n-i+1)$, which is what we set out to show.

Now we will prove $n \log n$ is $O(\log n!)$.

$$\begin{aligned} n \log n &= \log n + \log n + \log n + \cdots + \log n \\ &\leq \log(1 \cdot n) + \log(2 \cdot (n-1)) + \log(3 \cdot (n-2)) + \cdots + \log(n \cdot 1) \\ &= [\log 1 + \log n] + [\log 2 + \log(n-1)] + [\log 3 + \log(n-2)] + \cdots + [\log n + \log 1] \\ &= 2 \log 1 + 2 \log 2 + 2 \log 3 + \cdots + 2 \log n \\ &= 2(\log 1 + \log 2 + \log 3 + \cdots + \log n) \\ &= 2 \log(1 \cdot 2 \cdot 3 \cdot \cdots \cdot n) \\ &= 2 \log n! \end{aligned}$$

Therefore $n \log n$ is $O(\log n!)$ with $C = 2$ and $k = 1$ as witnesses.