# Fibonacci Numbers

The Fibonacci numbers, $f_0$, $f_1$, $f_2, \ldots$, are defined recursively by the equations $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2},$$

for $n = 2, 3, 4, \ldots$. Fibonacci numbers form a sequence $0, 1, 1, 2, 3, 5, 8, 13, 21 \ldots$. Using Strong Induction we can prove that

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for $n = 0, 1, 2, \ldots$.

*Proof:* Let $P(n)$ be the statement that this formula is true, where $n \geq 0$.
(Base Case) We first show $P(0)$ and $P(1)$ are true, i.e. that the formula is valid when $n = 0$ and $n = 1$. If $n = 0$, then

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^0 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^0 = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0 = f_0,$$

and if $n = 1$, then

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^1 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^1 = \frac{1+\sqrt{5}}{2\sqrt{5}} - \frac{1-\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 = f_1.$$

(Inductive Step) Let $k \geq 1$ and suppose $P(j)$ is true for all $0 \leq j \leq k$. Since $k \geq 1$, then $k - 1 \geq 0$ and so in particular we are assuming that $P(k)$ and $P(k-1)$ are both true. We must show $P(k+1)$ is true. In the following we will make use of the observation that

$$\left(\frac{1\pm\sqrt{5}}{2}\right)^2 = \frac{1 \pm 2\sqrt{5} + 5}{4} = \frac{6 \pm 2\sqrt{5}}{4} = \frac{3 \pm \sqrt{5}}{2} = \frac{1 \pm \sqrt{5}}{2} + 1.$$

From the recursive definition of the Fibonacci numbers and from our inductive hypothesis we get

$$\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \\
&= \left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k\right] + \left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2} + 1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2} + 1\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right] \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}.
\end{aligned}$$

Thus $P(k+1)$ is true. By Strong Induction we have proven that the formula is valid for all $n \geq 0$.

**Example 1:** Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for all positive integers $n$.

*Proof:* We can use the Principle of Mathematical Induction here; we do not need Strong Induction.
(Base Case) If $n = 1$, then $f_1^2 = 1^1 = 1$ and $f_1 f_2 = 1 \cdot 1 = 1$ and so $f_1^2 = f_1 f_2$.
(Inductive Step) Suppose $f_1^2 + f_2^2 + \cdots + f_k^2 = f_k f_{k+1}$, where $k \geq 1$. Then

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2 &= (f_1^2 + f_2^2 + \cdots + f_k^2) + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1}^2 \qquad \text{by the inductive hypothesis} \\ &= f_{k+1}(f_k + f_{k+1}) \qquad \text{by factoring} \\ &= f_{k+1} f_{k+2} \qquad \text{by the recursive definition of the Fibonacci numbers.} \end{aligned}$$

Therefore by the Principle of Mathematical Induction $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for all $n \geq 1$.

**Example 2:** Prove that every positive integer $n$ can be written as the sum of one or more distinct Fibonacci numbers.

Before proving this statement, we note that every Fibonacci number can itself be written as the sum of one or more (in this case just one) Fibonacci numbers. The problem therefore involves proving that non-Fibonacci numbers can also be so written. Here is a list of a few such integers. Note that these representations are not unique; for example $6 = 5 + 1$ and $6 = 3 + 2 + 1$.

| | | | |
|---|---|---|---|
| $1 = 1$ | $2 = 2$ | $3 = 3$ | $4 = 3 + 1$ |
| $5 = 5$ | $6 = 5 + 1$ | $7 = 5 + 2$ | $8 = 8$ |
| $9 = 8 + 1$ | $10 = 8 + 2$ | $11 = 8 + 3$ | $12 = 8 + 3 + 1$ |

Suppose we wanted to show that the number 62 (which is not a Fibonacci number) could be written as a sum of distinct Fibonacci numbers. We could begin by looking for the largest Fibonacci number that is less than 62, in this case 55, and writing 62 as $55 + 7$. Since the difference between 62 and 55 (i.e. 7) must be smaller than 55, then if we know how to decompose it into a sum of distinct Fibonacci numbers ($7 = 5 + 2$), then we can add 55 to this sum and write $62 = 55 + 5 + 2$, where we have now expressed 62 as a sum of distinct Fibonacci numbers. We will use this idea when establishing the inductive step of our induction argument. We need Strong Induction because we are using the fact that 7, rather than 61, can be expressed as a sum of distinct Fibonacci numbers in order to prove the same about 62.

*Proof:*
(Base Case) If $n = 1$, then $1 = f_1$.
(Inductive Step) Suppose that every integer $1, 2, 3, \ldots, k$ can be written as the sum of one or more distinct Fibonacci numbers, where $k \geq 1$. Consider the integer $n = k + 1$. If it is a Fibonacci number, then there is nothing further to show, so suppose it is not. Let $f_j$ denote the largest Fibonacci number that is less than $k + 1$. In other words $f_j < k + 1 < f_{j+1}$. Subtracting $f_j$ gives

$$0 < (k + 1) - f_j < f_{j+1} - f_j = f_{j-1} \leq f_j < k + 1.$$

The integer $(k + 1) - f_j$ belongs to the set $\{1, 2, 3, \ldots, k\}$ and so by our inductive hypothesis it can be written as the sum of one or more distinct Fibonacci numbers. Since $(k + 1) - f_j$ is also less than $f_j$, then $f_j$ is not part of this sum. By adding $f_j$ to $(k + 1) - f_j$ one gets $k + 1$ expressed as a sum of distinct Fibonacci numbers.

Therefore by Strong Induction we have proven that every positive integer $n$ can be written as the sum of one or more distinct Fibonacci numbers.