Divide-and-Conquer Theorems

Theorem 1: Let f satisfy the recurrence relation

$$f(n) = af(n/b) + c,$$

whenever b divides n, where $a \geq 1$ is a real number, $b > 1$ is an integer and $c > 0$ is a real number. If f is an increasing function, then

$$f(n) \text{ is } \begin{cases} O(\log n), & \text{if } a = 1, \\ O(n^{\log_b a}), & \text{if } a > 1. \end{cases}$$

Furthermore, if $n = b^k$, where $k \geq 1$ is an integer, then

$$f(n) = \begin{cases} f(1) + c \log_b n, & \text{if } a = 1, \\ \left[f(1) + \dfrac{c}{a-1} \right] n^{\log_b a} - \dfrac{c}{a-1}, & \text{if } a > 1. \end{cases}$$

Theorem 2 (Master Theorem): Let f be an increasing function satisfying the recurrence relation

$$f(n) = af(n/b) + cn^d,$$

whenever $n = b^k$, where $k \geq 1$ is an integer, $a \geq 1$ is a real number, $b > 1$ is an integer, $c > 0$ is a real number and $d \geq 0$ is a real number. Then

$$f(n) \text{ is } \begin{cases} O(n^d), & \text{if } a < b^d, \\ O(n^d \log n), & \text{if } a = b^d, \\ O(n^{\log_b a}), & \text{if } a > b^d. \end{cases}$$