Cardinality of Sets

The **cardinality** of a set A, denoted $|A|$, is a measure of the size of the set. For finite sets, the cardinality is simply the number of elements in the set. For example, if $A = \{a, b, c\}$, then $|A| = 3$.

To extend the notion of cardinality to infinite sets we start by defining the notion of comparing sets. In so doing we can compare different "sizes" or "levels" of infinity.

The sets of integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, and real numbers $\mathbb{R}$ are all infinite. Moreover $\mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{Q} \subset \mathbb{R}$. However, as we will soon discover, functionally the cardinality of $\mathbb{Z}$ and $\mathbb{Q}$ are the same, i.e. $|\mathbb{Z}| = |\mathbb{Q}|$, and yet both sets have a smaller cardinality than $\mathbb{R}$, i.e. $|\mathbb{Z}| < |\mathbb{R}|$.

Notation: The cardinality of the set of positive integers $\mathbb{Z}^+$ is denoted $\aleph_0$, pronounced *aleph null*, where $\aleph$ is the first letter of the Hebrew alphabet, i.e. $|\mathbb{Z}^+| = \aleph_0$. $\square$

Notation: The cardinality of the set of real numbers $\mathbb{R}$ is denoted $\mathfrak{c}$ (a lowercase Fraktur script "c"), i.e. $|\mathbb{R}| = \mathfrak{c}$. $\square$

Definition: Sets A and B are said to have the same cardinality, written $|A| = |B|$, if there is a one-to-one correspondence (i.e. bijection) between them. If not, then we write $|A| \neq |B|$. $\square$

Example: The sets $A = \{a, b, c\}$ and $B = \{x, y, z\}$ have the same cardinality (in this case 3) since the function $f : A \to B$ defined by $f(a) = x$, $f(b) = y$ and $f(c) = z$ is clearly a bijection. $\square$

Example: The sets of positive even integers $E^+ = \{2, 4, 6, 8, \ldots\}$ and all positive integers $\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$ have the same cardinality, i.e. $|E^+| = |\mathbb{Z}^+|$, despite the fact that E^+ is a proper subset of $\mathbb{Z}^+$. To show this we exhibit a bijection from $\mathbb{Z}^+$ to E^+ (or vice versa). Consider the function $f : \mathbb{Z}^+ \to E^+$ defined by $f(n) = 2n$. This function is one-to-one since if $f(n_1) = f(n_2)$ for $n_1, n_2 \in \mathbb{Z}^+$, then $2n_1 = 2n_2$ and hence $n_1 = n_2$. The function is onto since if $y \in E^+$, then $y = 2k$ for some $k \in \mathbb{Z}^+$ (by the definition of *even* and since y is positive) from which we find $f(k) = y$. $\square$

For two sets to have the same cardinality, this essentially means that the elements of one set can be "renamed" or "converted" into those of the other set (and vice versa), thereby establishing the fact the sets are the same "size."

Definition: Given sets A and B, the cardinality of A is said to be less than or the same as that of B, written $|A| \leq |B|$, if there is a one-to-one function from A to B. If $|A| \leq |B|$ but $|A| \neq |B|$, then the cardinality of A is said to be less than that of B, written $|A| < |B|$. $\square$

Example: If $A = \{a, b, c\}$ and $B = \{w, x, y, z\}$, then $|A| \leq |B|$ since the function $f : A \to B$ defined by $f(a) = x$, $f(b) = y$ and $f(c) = z$ is clearly one-to-one. There are no onto functions from A to B, so in this case $|A| < |B|$. $\square$

Definition: If a set A is finite or if $|A| = \aleph_0$ (i.e. A has the same cardinality as $\mathbb{Z}^+$), then A is said to be **countable**. Otherwise it is said to be **uncountable**. $\square$

The sets $A = \{a, b, c\}$ and $B = \{w, x, y, z\}$ are countable since they are finite. The infinite set E^+ (and of course the set $\mathbb{Z}^+$ itself) is also countable since we previously showed that $|E^+| = |\mathbb{Z}^+|$.

Note: An infinite set is countable if, and only if, its elements can be listed in a sequence (indexed by the positive integers). $\square$

Example: The integers $\mathbb{Z}$ is countable since the elements can be listed in a sequence such as

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots,$$

where the pattern by which the sequence is constructed is clear. Another way to view this one-to-one correspondence between $\mathbb{Z}^+$ and $\mathbb{Z}$ is as follows:

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & -4 & \end{array}$$

This bijection can also be written explicitly using a piecewise function

$$f(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even,} \\ -\dfrac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

$\square$

Theorem: Any subset of a countable set is countable.
Proof: If A is a countable set, we can list its elements $a_1, a_2, a_3, \ldots, a_n, \ldots$, possibly ending after a finite number of terms. If $B \subseteq A$, then we can list the elements of B (if any) in the same order in which they appear in A. Therefore B is countable. $\blacksquare$

Theorem: If A and B are countable sets, then so is $A \cup B$.
Proof: Suppose A and B are countable sets. Then we can list the elements of A and B as $a_1, a_2, a_3, \ldots, a_n, \ldots$, and $b_1, b_2, b_3, \ldots, b_m, \ldots$, respectively, where these sequences may end after a finite number of terms. One can list the elements of $A \cup B$ by alternating terms according to $a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_n, b_n, \ldots$ except for omitting elements already appearing earlier in the list (if $A \cap B \neq \varnothing$) and if either the list for A or B stops (because one or both sets is finite), then we do not list the nonexistent terms. Therefore $A \cup B$ is countable. $\blacksquare$

Theorem: The set of positive rational numbers $\mathbb{Q}^+$ is countable.
Proof: To be done in class....

Corollary: The set of rational numbers $\mathbb{Q}$ is countable, i.e. $|\mathbb{Q}| = \aleph_0$.
Proof: Since $\mathbb{Q}^+$ is countable, we can list its elements in a sequence, $\mathbb{Q}^+ = \{q_1, q_2, q_3, \ldots, q_n, \ldots\}$. The rational numbers can therefore also be listed in a sequence

$$\mathbb{Q} = \{0, q_1, -q_1, q_2, -q_2, q_3, -q_3, \ldots, q_n, -q_n, \ldots\},$$

proving it is countable. ∎

Theorem: The union of a countable number of countable sets is countable.
Proof: Suppose $A_1, A_2, A_3, \ldots$ are countable sets. Because A_i is countable, its elements can be listed in a sequence $a_{i1}, a_{i2}, a_{i3}, \ldots$. The elements of the set $\cup_{i=1}^{n} A_i$ (or $\cup_{i=1}^{\infty} A_i$) can be listed by listing all terms a_{ij} with $i + j = 2$, then all terms a_{ij} with $i + j = 3$, then all terms a_{ij} with $i + j = 4$ and so on (similar to how we listed the positive rational numbers). Therefore the union of these countable sets is countable. ∎

Theorem: The set of real numbers $\mathbb{R}$ is uncountable.
Proof (by contradiction): To be done in class....

Corollary: The set of irrational numbers $\mathbb{I}$ is uncountable.
Proof: If $\mathbb{I}$ were countable, then $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ would be countable (a union of countable sets), which it is not. Therefore $\mathbb{I}$ is uncountable. $\blacksquare$

Theorem: If $|A| = |B|$ for sets A and B, then $|A| \le |B|$ and $|B| \le |A|$.
Proof: This proof is relatively easy and is left to the student. Keep in mind what this theorem is saying, namely that if there exists a one-to-one correspondence $f : A \to B$, then there exists a one-to-one function from A to B and there exists a one-to-one function from B to A. $\square$

Schröder-Bernstein Theorem: If $|A| \le |B|$ and $|B| \le |A|$ for sets A and B, then $|A| = |B|$.
Proof: This is difficult to prove in the case of uncountable sets. $\square$

Recall that if $\mathcal{P}(A)$ denotes the power set of a finite set A, then $|\mathcal{P}(A)| = 2^{|A|}$. In particular $|A| < |\mathcal{P}(A)|$. This relationship extends to infinite sets as stated in the following theorem.

Theorem: $|A| < |\mathcal{P}(A)|$ for any set A. $\square$

Finally, here are some other interesting theorems regarding the cardinality of power sets.

Theorem: The power set of an infinite, countable set is uncountable. $\square$

Theorem: $|\mathcal{P}(\mathbb{Z}^+)| = |\mathbb{R}| = \mathfrak{c}$. $\square$