Pigeonhole Principle

Pigeonhole Principle: Let $k \in \mathbb{Z}^+$. If $k+1$ or more objects are placed into k boxes, then at least one box must contain two or more objects.

Proof (by contraposition): Suppose the conclusion were false. Then each of the k boxes has at most one object and so there are at most k objects in total. Therefore it is not true that $k+1$ or more objects were placed into the boxes.

Example: If 10 or more pigeons are occupying 9 pigeonholes, then at least one of the pigoneholes must contain two or more pigeons.

Example: If 21 students are writing a test in a room having only 20 desks, then at least two people are sharing the same desk.

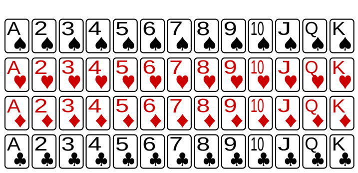
Example: Among a group of 400 people at least two have the same birthday (since there are only 366 possible birthdays).

Exercise: How many times must you roll a standard 6-sided die so you are guaranteed to roll a number twice?

Generalized Pigeonhole Principle: If N objects are placed into k boxes, then at least one box contains at least $\lceil N/k \rceil$ objects.

Example: Among a group of 400 people at least $\lceil 400/12 \rceil = 34$ were born in the same month. Note that if as many as 33 people were born in each of the 12 months, then that totals only 396 people.

Exercise: How many cards from a standard deck of 52 playing cards must be selected to guarantee that at least 5 cards are from the same suit?



Exercise: How many numbers must be selected from the set $S = \{1, 2, 3, \ldots, 100\}$ to guarantee that at least one pair of these numbers

(a) adds up to 101?

(b) are consecutive integers?

Exercise: Prove that there exist two distinct powers of 3 whose difference is divisible by 1999.