One-to-One and Onto

Definition: Let $f : A \to B$ be a function. Then f is said to be

1. **one-to-one** or **injective** if

$$\forall x_1 \in A \; \forall x_2 \in A \; \Big((f(x_1) = f(x_2)) \to (x_1 = x_2) \Big),$$

   or equivalently,

$$\forall x_1 \in A \; \forall x_2 \in A \; \Big((x_1 \neq x_2) \to (f(x_1) \neq f(x_2)) \Big),$$

2. **onto** or **surjective** if

$$\forall y \in B \; \exists x \in A \; (f(x) = y),$$

   or equivalently,

$$f(A) = B,$$

   i.e. the range equals the codomain.

3. **bijective** if f is one-to-one and onto; in this case f is called a **bijection** or a **one-to-one correspondence**.

Examples: Determine whether the following functions are one-to-one or onto.

1. $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.
   (a) f is not one-to-one since -3 and 3 are in the domain and $f(-3) = 9 = f(3)$.
   (b) f is not onto since, for example, there is no real number that maps to -2 in the codomain. If $x^2 = -2$, then $x = \pm\sqrt{-2}$, which is not real.

2. $f : \mathbb{Z}^+ \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$.
   (a) f is one-to-one since if x_1 and x_2 are positive integers and $\sqrt{x_1} = \sqrt{x_2}$, then $x_1 = x_2$ by squaring both sides.
   (b) f is not onto since, for example, there is no positive integer that maps to $1/2$ in the codomain. If $\sqrt{x} = 1/2$, then $x = 1/4$, which is not a positive integer.

3. $f : \mathbb{Z} \to \mathbb{Z}^+$ be defined by $f(x) = |x| + 1$.
   (a) f is not one-to-one since -2 and 2 are in the domain and $f(-2) = 3 = f(2)$.
   (b) f is onto since if y is any positive integer in the codomain, then $x = y - 1$ is a (nonnegative) integer in the domain whereby $f(x) = |x| + 1 = x + 1 = (y-1) + 1 = y$. Note that the (nonpositive) integer $x = -y + 1$ could instead have been used.

4. $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - 4$.
   (a) f is one-to-one since if x_1 and x_2 are real numbers and $x_1^3 - 4 = x_2^3 - 4$, then $x_1^3 = x_2^3$ and so $x_1 = x_2$.
   (b) f is onto since if y is any real number in the codomain, then $x = \sqrt[3]{y+4}$ is a real number in the domain whereby $f(x) = (\sqrt[3]{y+4})^3 - 4 = (y + 4) - 4 = y$.