Images of Set Unions and Intersections

Theorem: Let $f : A \to B$ be a function. If $S \subseteq A$ and $T \subseteq A$, then

1. $f(S \cup T) = f(S) \cup f(T)$, and
2. $f(S \cap T) \subseteq f(S) \cap f(T)$.

Proof:

1. We prove equality by showing $f(S \cup T)$ and $f(S) \cup f(T)$ are subsets of each other.

   Suppose $y \in f(S \cup T)$. Then $y = f(x)$ for some $x \in S \cup T$. Since $x \in S \cup T$, then $x \in S$ or $x \in T$. Therefore $y \in f(S)$ or $y \in f(T)$, which implies $y \in f(S) \cup f(T)$. This proves $f(S \cup T) \subseteq f(S) \cup f(T)$.

   Now suppose $y \in f(S) \cup f(T)$. Then $y \in f(S)$ or $y \in f(T)$. Therefore $y = f(x)$ for some $x \in S$ or $y = f(x)$ for some $x \in T$. In either case $y = f(x)$ for some $x \in S \cup T$, which implies $y \in f(S \cup T)$. This proves $f(S) \cup f(T) \subseteq f(S \cup T)$.

   Since $f(S \cup T) \subseteq f(S) \cup f(T)$ and $f(S) \cup f(T) \subseteq f(S \cup T)$, then $f(S \cup T) = f(S) \cup f(T)$.

2. Suppose $y \in f(S \cap T)$. Then $y = f(x)$ for some $x \in S \cap T$. Since $x \in S \cap T$, then $x \in S$ and $x \in T$. Therefore $y \in f(S)$ and $y \in f(T)$, which implies $y \in f(S) \cap f(T)$. This proves $f(S \cap T) \subseteq f(S) \cap f(T)$.

Note: In general $f(S \cap T) \neq f(S) \cap f(T)$ unless f is one-to-one.

Example: Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = n^2$. If

$$S = \{0, 1, 2, 3\} \text{ and } T = \{-2, -1, 0, 1\},$$

then

$$\begin{aligned}
f(S) &= \{0, 1, 4, 9\} \\
f(T) &= \{0, 1, 4\} \\
S \cup T &= \{-2, -1, 0, 1, 2, 3\} \\
S \cap T &= \{0, 1\} \\
f(S \cup T) &= \{0, 1, 4, 9\} \\
f(S) \cup f(T) &= \{0, 1, 4, 9\} \\
f(S \cap T) &= \{0, 1\} \\
f(S) \cap f(T) &= \{0, 1, 4\}.
\end{aligned}$$

Here $f(S \cup T) = f(S) \cup f(T)$ and $f(S \cap T) \subseteq f(S) \cap f(T)$, yet $f(S \cap T) \neq f(S) \cap f(T)$.