Special Types of Series

1. **Geometric Series**

   A geometric series is a series whose terms form a geometric sequence. If a is the first term and r is the common ratio (where both a and r are nonzero constants), then a geometric series has the form

   $$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n + \cdots .$$

   Note that indexing of a geometric series usually starts at $n = 0$. The partial sums are

   $$\begin{aligned} S_1 &= a \\ S_2 &= a + ar \\ S_3 &= a + ar + ar^2 \\ S_4 &= a + ar + ar^2 + ar^3 \\ &\vdots \\ S_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, \text{ if } r \neq 1. \end{aligned}$$

   If $|r| \geq 1$, then the geometric series diverges according to the n^{th} Term Test. On the other hand if $0 < |r| < 1$, then $r^n \to 0$ as $n \to \infty$. Therefore, the geometric series converges and its sum is

   $$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \qquad \text{if } 0 < |r| < 1.$$

2. **Telescoping Series**

Telescoping series usually have the form

$$\sum_{n=1}^{\infty}(b_n - b_{n+1}) = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots + (b_n - b_{n+1}) + \cdots .$$

The partial sums of such a series are given by

$$\begin{aligned}
S_1 &= b_1 - b_2 \\
S_2 &= (b_1 - b_2) + (b_2 - b_3) = b_1 - b_3 \\
S_3 &= (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) = b_1 - b_4 \\
S_4 &= (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) = b_1 - b_5 \\
&\vdots \\
S_n &= (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots + (b_n - b_{n+1}) = b_1 - b_{n+1}.
\end{aligned}$$

This telescoping series diverges if $\lim_{n\to\infty} b_{n+1}$ (or equivalently $\lim_{n\to\infty} b_n$) does not exist. But if this limit does exist, then the series converges and its sum is

$$\sum_{n=1}^{\infty}(b_n - b_{n+1}) = b_1 - \lim_{n\to\infty} b_{n+1}.$$

Another type of telescoping series is one of the form

$$\sum_{n=1}^{\infty}(b_n - b_{n+2}) = (b_1 - b_3) + (b_2 - b_4) + (b_3 - b_5) + \cdots + (b_n - b_{n+2}) + \cdots .$$

The n^{th} partial sum of this series (for $n \geq 2$) is

$$S_n = b_1 + b_2 - b_{n+1} - b_{n+2} = (b_1 + b_2) - (b_{n+1} + b_{n+2}).$$

This series diverges if $\lim_{n\to\infty}(b_{n+1} + b_{n+2})$ does not exist. Otherwise the series converges and its sum is

$$\sum_{n=1}^{\infty}(b_n - b_{n+2}) = b_1 + b_2 - \lim_{n\to\infty}(b_{n+1} + b_{n+2}).$$

3. p**-series**

A p-series has the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots ,$$

where $p > 0$ is constant.

A p-series converges if $p > 1$ and it diverges to infinity if $0 < p \leq 1$. The Integral Test can be used to verify this. If $p > 1$, then it is a nontrivial problem (beyond the scope of our course) to determine what it converges to. Here are a few particular cases

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$
$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Interestingly, if p is an odd integer, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \neq \frac{\pi^p}{m}$ for any integer m.

If $p = 1$, then one gets a very special type of p-series known as the **harmonic series**,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots .$$

This series diverges to infinity, albeit very slowly.