Properties of Series

If A and B are real numbers and

$$\sum_{n=1}^{\infty} a_n = A \quad \text{and} \quad \sum_{n=1}^{\infty} b_n = B,$$

(i.e. both series converge), then

1. $\sum_{n=1}^{\infty} ca_n = cA$, for any real number c, and
2. $\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$.

Note that in general $\sum_{n=1}^{\infty} a_n b_n \neq AB$ since

$$AB = \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n = (a_1 + a_2 + a_3 + a_4 + \cdots)(b_1 + b_2 + b_3 + b_4 + \cdots),$$

and expanding this product produces

$$\begin{aligned} &\underline{a_1 b_1} + a_2 b_1 + a_1 b_2 + a_3 b_1 + \underline{a_2 b_2} + a_1 b_3 + a_4 b_1 + a_3 b_2 + a_2 b_3 + a_1 b_4 \\ &\qquad + a_5 b_1 + a_4 b_2 + \underline{a_3 b_3} + a_2 b_4 + a_1 b_5 + a_6 b_1 + a_5 b_2 + a_4 b_3 + a_3 b_4 + a_2 b_5 + a_1 b_6 + \cdots, \end{aligned}$$

which contains numerous cross-terms of the form $a_i b_j$, where $i \neq j$, besides the underlined terms that make up the series $\sum_{n=1}^{\infty} a_n b_n$.

To illustrate this last point, note that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ and so $\left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right)^2 = 1$.
However, $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} \right)^2 = \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3} \neq 1$.