CAMOSUN

Mathematics 101 Test #3A

Name: SOLUTIONS

Instructor: George Ballinger

Section: ___________

Term: Winter, 2015

The Sharp EL-531 calculator may be used on this test.
Show all of your work in the space provided.
The number of marks for each question is indicated in brackets.
Give exact answers (no decimals) unless told otherwise.

Mark: $\frac{}{25}$

1. Determine whether the given series converges or diverges and identify which test allows you to reach your conclusion. Assume $b_n > 0$ for all positive integers *n*.

   (a) If $\lim_{n\to\infty} b_n = 2$, then does $\sum_{n=1}^{\infty} b_n$ converge or diverge? diverges by n^{th} term test

[1]

   (b) If $\lim_{n\to\infty} b_n = 2$, then does $\sum_{n=1}^{\infty} (b_n - b_{n+1})$ converge or diverge? Converges by telescoping series test

[1]

   (c) If $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 2$, then does $\sum_{n=1}^{\infty} b_n$ converge or diverge? diverges by ratio test

[1]

2. Consider the convergent series $\sum_{n=1}^{\infty} e^{-n}$.

   (a) Using your calculator, approximate the sum of the series by adding the first four terms of the series. Round your answer to four decimal places.

[1] $e^{-1}+e^{-2}+e^{-3}+e^{-4} \approx 0.5713$

   (b) Verify that the series satisfies the conditions of the integral test. Then use the remainder formula from the integral test to estimate the error in your approximation in part (a) and identify an interval in which the exact sum of the series must lie. Round all values to four decimal places.

$f(x) = e^{-x}$ is continuous, positive and decreasing for $x \geq 1$.

$$R_4 < \int_4^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_4^{b} e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_4^b = \lim_{b\to\infty} \left[-e^{-b}+e^{-4}\right]$$

$$= e^{-4} \approx 0.0183 \text{ (max. error)}$$

[3]

$$\therefore \quad 0.5713 < \sum_{n=1}^{\infty} e^{-n} < 0.5713 + 0.0183 = 0.5896$$

   (c) What type of series is this? Find the **exact** value of the sum of the series.

Geometric series

[2] $a = e^{-1}$, $r = e^{-1}$

$$\therefore \sum_{n=1}^{\infty} e^{-n} = \frac{e^{-1}}{1-e^{-1}} = \frac{1}{e-1} \approx 0.5820$$

3. Determine whether the series converges conditionally, converges absolutely, or diverges. Identify which test(s) you are using and show that all of the conditions of the test(s) are satisfied.

$\sum_{n=1}^{\infty} \frac{3(-1)^n}{3n-2}$

Alt. series

$\lim_{n\to\infty} \frac{3}{3n-2} = 0$ and $\frac{3}{3(n+1)-2} \le \frac{3}{3n-2}$ for $n \ge 1$

$\therefore$ Series Converges by AST

Consider $\sum_{n=1}^{\infty} \frac{3}{3n-2}$. This series diverges by comparing it * (using DCT or LCT) with divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

[4]

DCT

$0 < \frac{1}{n} < \frac{1}{n-2/3} = \frac{3}{3n-2}$ for $n \ge 1$ ✓

OR

LCT

$\lim_{n\to\infty} \frac{\frac{3}{3n-2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n}{3n-2} = \lim_{n\to\infty} \frac{3}{3-\frac{2}{n}} = 1$ ✓

(finite & positive)

(* or using integral test)

$\therefore \sum_{n=1}^{\infty} \frac{3(-1)^n}{3n-2}$ Converges Conditionally

4. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3x)^n}{\sqrt{n}}$. Be sure to check the endpoints.

Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(3x)^{n+1}}{\sqrt{n+1}}}{\frac{(3x)^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{(3x)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(3x)^n} \right| = \lim_{n\to\infty} |3x| \sqrt{\frac{n}{n+1}}$$

$$= |3x| \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 3|x| \cdot 1 = 3|x|$$

$\therefore$ Absolute Convergence when $3|x| < 1$

i.e. $|x| < \frac{1}{3}$ or $-\frac{1}{3} < x < \frac{1}{3}$

[5]

Endpoints

check $x = -\frac{1}{3}$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{\sqrt{n}} = -\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges (p-series, $p = \frac{1}{2} < 1$)

check $x = \frac{1}{3}$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

Converges by AST since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and

$\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}}$ for $n \ge 1$

and series alternates

$\therefore$ interval of Convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right]$

5. Find the third Taylor polynomial, $P_3(x)$, for $f(x)=\sqrt{x}$ centered at $c=1$.

[2]

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$1/2$
2	$-\frac{1}{4}x^{-3/2}$	$-1/4$
3	$\frac{3}{8}x^{-5/2}$	$3/8$

$$P_3(x) = 1 + \tfrac{1}{2}(x-1) + \frac{-1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3$$

$$= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3$$

6.

(a) Use the Maclaurin series for e^x to find a power series for $e^{-\frac{1}{3}x^3}$. Express your answer using Σ-notation.

[1]

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n \quad \text{for } -\infty < x < \infty$$

$$e^{-\frac{1}{3}x^3} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\tfrac{1}{3}x^3\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,3^n} x^{3n}$$

(b) Use the power series in part (a) to express (using Σ-notation) $\int_0^1 e^{-\frac{1}{3}x^3}\,dx$ as an infinite series.

[2]

$$\int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,3^n} x^{3n}\,dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,3^n(3n+1)} x^{3n+1}\right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,3^n(3n+1)}$$

(c) Use the first three terms of the series in part (b) to approximate $\int_0^1 e^{-\frac{1}{3}x^3}\,dx$ and estimate the size of the error by using the Alternating Series Remainder theorem.

[2]

$$\int_0^1 e^{-\frac{1}{3}x^3}\,dx = \underbrace{1 - \frac{1}{12} + \frac{1}{126}} - \frac{1}{1620} + \cdots$$

Alt. Series satisfying ASR Conditions

$$\approx \frac{233}{252} \approx 0.92460$$

$$|\text{Max error}| \le \frac{1}{1620} \approx 0.00062$$