CAMOSUN

Mathematics 101 Test #3B **Name:** SOLUTIONS

Instructor: George Ballinger **Section:** ___________

Term: Winter, 2016

The Sharp EL-531 calculator may be used on this test.
Show all of your work in the space provided.
The number of marks for each question is indicated in brackets.
Give exact answers (no decimals) unless told otherwise.

Mark: $\overline{\quad}$ 25

1. Consider the sequence $\{a_n\} = \frac{3}{2}, \frac{4}{4}, \frac{5}{6}, \frac{6}{8}, \frac{7}{10}, \ldots$

   (a) Find a formula, a_n, for the n^{th} term of the sequence.

[1] $a_n = \frac{n+2}{2n}$

   (b) Determine whether the sequence $\{a_n\}$ converges or diverges. If it converges, find its limit. If it diverges, determine whether it diverges to infinity, negative infinity, or neither.

[1] $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+2}{2n} = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{n}\right) = \frac{1}{2}$ $\therefore$ converges

   (c) Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges and identify the test you are using.

[1] Series diverges by n^{th} term test since $\lim_{n\to\infty} a_n \neq 0$.

   (d) Would the 100^{th} partial sum, $S_{100} = a_1 + a_2 + a_3 + \cdots + a_{100}$, of the series $\sum_{n=1}^{\infty} a_n$ provide a reasonable approximation for the sum of the series? Why or why not?

No. This series diverges to ∞. It doesn't have a finite sum that a partial sum could approximate.

[1]

2. Find the **exact** sum of each of the following convergent series.

   (a) $\sum_{n=2}^{\infty}\left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \cdots = \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \frac{1/9}{1-1/3} = \frac{1/9}{2/3} = \frac{1}{6}$

(geometric series with $a = 1/9$ and $r = 1/3$)

[2]

OR $\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^0 - \left(\frac{1}{3}\right)^1 = \frac{1}{1-1/3} - 1 - \frac{1}{3} = \frac{3}{2} - \frac{4}{3} = \frac{1}{6}$

   (b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} 10^n = \sum_{n=0}^{\infty} \frac{1}{n!}(-10)^n = e^{-10}$

[1]

3. Find a geometric power series (using Σ-notation) for $f(x)=\dfrac{1}{2x-5}$ centered at $c=3$ and determine its interval of convergence.

$$f(x)=\frac{1}{-5+2(x-3+3)}=\frac{1}{-5+2(x-3)+6}=\frac{1}{1+2(x-3)}=\frac{1}{1-(-2)(x-3)}$$

$$=\sum_{n=0}^{\infty}\left[(-2)(x-3)\right]^n=\sum_{n=0}^{\infty}(-2)^n(x-3)^n$$

[4] Converges iff $|(-2)(x-3)|<1$

$$2|x-3|<1$$
$$|x-3|<\tfrac{1}{2}$$
$$-\tfrac{1}{2}<x-3<\tfrac{1}{2}$$
$$\tfrac{5}{2}<x<7/2$$

Interval of Convergence is $\left(\frac{5}{2},\frac{7}{2}\right)$

4. Verify that the conditions of the integral test are satisfied by the series and use the integral test to determine whether the series converges or diverges.

$\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ Let $f(x)=\frac{1}{x^2+1}$. $f(x)$ is continuous, positive and decreasing for $x\geq 1$.

$$\int_1^{\infty}\frac{1}{x^2+1}\,dx=\lim_{b\to\infty}\int_1^b\frac{1}{1+x^2}\,dx=\lim_{b\to\infty}\left[\arctan x\right]_1^b$$

[3] $$=\lim_{b\to\infty}\left[\arctan b-\arctan 1\right]=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$ Converges.

Since integral converges, then so does series by integral test.

5. Use a comparison test to determine whether the series converges or diverges. Identify which comparison test you are using and show that all of the conditions of the test are satisfied.

$\underbrace{\sum_{n=1}^{\infty}\frac{n}{n^2+2n+3}}_{>0 \text{ for } n\geq 1}$ Compare with diverging harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ using LCT (can't use DCT since $\frac{n}{n^2+2n+3}<\frac{1}{n}$).

[3] $$\lim_{n\to\infty}\frac{\frac{n}{n^2+2n+3}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^2}{n^2+2n+3}=\lim_{n\to\infty}\frac{1}{1+\frac{2}{n}+\frac{3}{n^2}}=1$$

Since limit is finite and positive, then by LCT this series also diverges.

6. Determine whether the series converges conditionally, converges absolutely, or diverges. Identify which tests you are using and show that all of the conditions of the tests are satisfied.

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$$

alt. series converges by AST since $\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$ and $\frac{1}{(n+1)^{1/3}} \leq \frac{1}{n^{1/3}}$ for $n \geq 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is a diverging p-series since $p = \frac{1}{3} \leq 1$.

[3] $\therefore \sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt[3]{n}}$ Converges Conditionally.

7. Use the power series $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ to express $\int_0^{1/2} x \arctan x \, dx$ as a series. Then use the first two nonzero terms of the series to approximate the value of $\int_0^{1/2} x \arctan x \, dx$ and estimate the size of the error by using the Alternating Series Remainder theorem.

$$\int_0^{\frac{1}{2}} x \arctan x \, dx = \int_0^{\frac{1}{2}} x \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \, dx = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+2} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} x^{2n+3} \Big|_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)\, 2^{2n+3}}$$

alt. series satisfies conditions of ASR since $\lim_{n\to\infty} \frac{1}{(2n+1)(2n+3)\, 2^{2n+3}} = 0$ and $\frac{1}{(2n+3)(2n+5)\, 2^{2n+5}} \leq \frac{1}{(2n+1)(2n+3)\, 2^{2n+3}}$ for all $n \geq 1$.

$$= \frac{1}{(1)(3)(8)} - \frac{1}{(3)(5)(32)} + \frac{1}{(5)(7)(128)} - \cdots$$

[5]

$$= \underbrace{\frac{1}{24} - \frac{1}{480}} + \frac{1}{4480} - \cdots$$

$$\approx \frac{19}{480} \approx 0.039583333$$

$$|\text{Error}| = |R_2| \leq a_3 = \frac{1}{4480} \approx 0.000223214$$