CAMOSUN

**Mathematics 101 Test #3A** **Name:** SOLUTIONS

**Instructor: George Ballinger** **Section:** ____________

**Term: Winter, 2017**

**Mark:** $\frac{\phantom{00}}{25}$

> The Sharp EL-531 calculator may be used on this test.
>
> Show all of your work in the space provided.
>
> The number of marks for each question is indicated in brackets.
>
> Give exact answers (no decimals) unless told otherwise.

1. Consider the sequence $\{a_n\} = \frac{1(2!)}{3!}, \frac{2(3!)}{4!}, \frac{3(4!)}{5!}, \frac{4(5!)}{6!}, \frac{5(6!)}{7!}, \ldots$

   (a) Find and **simplify** a formula, $a_n$, for the $n^{\text{th}}$ term of the sequence.

   $$a_n = \frac{n(n+1)!}{(n+2)!} = \frac{n(n+1)!}{(n+2)(n+1)!} = \frac{n}{n+2}$$

   [2]

   (b) Determine whether the sequence $\{a_n\}$ converges or diverges. If it converges, find its limit. If it diverges, determine whether it diverges to infinity, negative infinity, or neither.

   $$\lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+\underbrace{\frac{2}{n}}_{\to 0}} = 1 \quad \therefore \text{ Converges}$$

   [1]

   (c) Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges and identify the test you are using.

   diverges by $n^{\text{th}}$ term test since $\lim_{n\to\infty} a_n \neq 0$.

   [1]

2. Find the **exact** sum of each of the following convergent series.

   (a) $\left(1-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{3}{5}\right)+\left(\frac{3}{5}-\frac{4}{7}\right)+\left(\frac{4}{7}-\frac{5}{9}\right)+\cdots+\left(\underbrace{\frac{n}{2n-1}}_{b_n}-\underbrace{\frac{n+1}{2n+1}}_{b_{n+1}}\right)+\cdots$ Telescoping series

   [2]

   $$= 1 - \lim_{n\to\infty} \frac{n+1}{2n+1}$$

   $$= 1 - \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{1}{n}} = 1 - \frac{1}{2} = \frac{1}{2}$$

   (b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}$

   [1]

   $$= \sin\frac{\pi}{2} = 1$$

3. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series where $0 < a_n \le b_n$ for all $n \ge 1$. Circle **all** of the statements that must be true. No justification is required.

DCT

[1]

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

(c) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(d) If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

(e) Either both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or they both diverge.

4. Determine whether each series converges or diverges. Identify which test you are using.

[1]

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by $p$-series test since $p = \frac{1}{2} \le 1$.

[1]

(b) $\sum_{n=0}^{\infty} \frac{1}{\pi^n} = \sum_{n=0}^{\infty} \left(\frac{1}{\pi}\right)^n$ Converges by geometric series test Since $r = \frac{1}{\pi} \approx 0.318$ and $|r| < 1$.

5. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

(a) Verify that the conditions of the integral test are satisfied by the series and use the integral test to determine whether the series converges or diverges.

[4]

Let $f(x) = \frac{1}{x \ln x}$ for $x \ge 2$. Then $f$ is continuous, positive and decreasing.

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \int_2^b \underbrace{\frac{1}{\ln x} \cdot \frac{1}{x}\,dx}_{\text{form } \frac{1}{u}du \text{ where } u = \ln x} = \lim_{b\to\infty} \Big[\ln|\ln x|\Big]_2^b$$

$$= \lim_{b\to\infty} \Big[\ln|\ln b| - \ln|\ln 2|\Big] = \infty.$$

Since integral diverges then so does the series by the integral test.

(b) The 2017th partial sum, $S_{2017}$, of this series is approximately 2.824156266. Explain whether or not this is a reasonable approximation for the sum of the series.

[1]

No, it's not. Series diverges to $\infty$ which no finite partial sum could approximate.

6. Find a geometric power series (using $\Sigma$-notation) for $f(x) = \dfrac{5}{2x+3}$ centred at $c = 1$ and determine its interval of convergence.

$$f(x) = \frac{5}{3+2x} = \frac{5}{3+2(x-1)+2} = \frac{5}{5+2(x-1)} = \frac{1}{1+\frac{2}{5}(x-1)} = \frac{1}{1-(-\frac{2}{5})(x-1)}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{2}{5}\right)^n (x-1)^n$$

Converges iff $\left|-\frac{2}{5}(x-1)\right| < 1$

[4]

$$|x-1| < 5/2$$

$$-\frac{5}{2} < x-1 < 5/2$$

$$-\frac{3}{2} < x < 7/2$$

$\therefore$ Interval of convergence is $\left(-\frac{3}{2}, \frac{7}{2}\right)$

7. Use the Maclaurin series for $\cos x$ to express $\int_0^1 \cos\sqrt{x}\,dx$ as a series. Then use the first three nonzero terms of the series to approximate the value of $\int_0^1 \cos\sqrt{x}\,dx$ and estimate the size of the error by using the Alternating Series Remainder theorem. Be sure to verify that the conditions of the Alternating Series Remainder theorem are satisfied.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for } -\infty < x < \infty$$

$$\therefore \cos\sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{x})^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n, \quad \text{for } x \geq 0$$

$$\int_0^1 \cos\sqrt{x}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\,(n+1)} x^{n+1}\Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\,(n+1)} = \underbrace{1 - \frac{1}{4} + \frac{1}{72}} - \frac{1}{2880} + \cdots$$

[6]

$$\approx \frac{55}{72} \approx 0.763889$$

Series is alternating, $\lim_{n\to\infty} \frac{1}{(2n)!\,(n+1)} = 0$

and $\frac{1}{[2n+2]!\,(n+2)} \leq \frac{1}{(2n)!\,(n+1)}$ for $n \geq 0$

$\therefore$ Series converges by AST and satisfies conditions of ASR

By ASR, $|\text{max error}| \leq \frac{1}{2880} \approx 0.000347$

(Note: exact value is $\int_0^1 \cos\sqrt{x}\,dx = 2\sin 1 + 2\cos 1 - 2 \approx 0.763546582$)