CAMOSUN

Mathematics 101 Test #3A **Name:** SOLUTIONS

Instructor: George Ballinger **Section:** ______

Term: Winter, 2018

Mark: $\overline{}$ / 25

The Sharp EL-531 calculator may be used on this test.
Show all of your work in the space provided.
The number of marks for each question is indicated in brackets.
Give exact answers (no decimals) unless told otherwise.

1. Find the limit of the sequence $\{a_n\}$ whose n^{th} term is given by $a_n = \frac{1}{n}\sin n$.

[2]

$\lim_{n\to\infty} \frac{\sin n}{n} = 0$ by Squeeze Theorem

Since $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$ for $n \ge 1$

and $\lim_{n\to\infty} -\frac{1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$

2. Find the **exact** sum of each of the following convergent series. Simplify your answers. Name the type of series appearing in part (c).

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{\pi}\right)^n = \frac{1}{1-(-\frac{1}{\pi})} = \frac{1}{1+\frac{1}{\pi}} = \frac{\pi}{\pi+1}$

geometric series
$r = -\frac{1}{\pi}$, $|r| < 1$

[1]

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\pi^n = \sum_{n=0}^{\infty} \frac{1}{n!}(-\pi)^n = e^{-\pi}$

[1]

(c) $\sum_{n=1}^{\infty} \left(\underbrace{\frac{n-1}{n}}_{b_n} - \underbrace{\frac{n}{n+1}}_{b_{n+1}}\right)$ Type of series: telescoping

[2]

$= b_1 - \lim_{n\to\infty} b_{n+1} = 0 - \lim_{n\to\infty} \frac{n}{n+1} = -\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = -1$

3. Answer the following True or False questions. No justification is required.

(a) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $f(x)$ is a positive, continuous and decreasing function for $x \ge 1$ and if $a_n = f(n)$ for all $n \ge 1$, then $\sum_{n=1}^{\infty} a_n = \int_1^{\infty} f(x)\,dx$.

(c) If the sequence $\{a_n\}$ converges, then $\lim_{n\to\infty}(a_n - a_{n+1}) = 0$.

(d) $\sum_{n=1}^{\infty} \cos n\pi$ is an alternating series.

[2] ANSWERS: (a) T (b) F (c) T (d) T

4. Determine whether each series converges or diverges. Identify which test you are using.

(a) $\sum_{n=1}^{\infty} \frac{1}{2}$ diverges by n^{th} term test since $\lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0$

[1]

(b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ $= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ Converges by p-series test since $p = 3/2 > 1$

[1]

5. Find and simplify a quadratic Taylor polynomial, $P_2(x)$, for $f(x) = \sqrt{x}$ centred at $c = 4$ and then use it to approximate $\sqrt{3.9}$. Express your answer in decimal form to as many decimals as your calculator will give you.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$x^{1/2}$	2
1	$\frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$	$\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2} = \frac{-1}{4x^{3/2}}$	$\frac{-1}{32}$

[3]

$$P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$$\therefore \sqrt{3.9} \approx P_2(3.9) = 2 + \frac{1}{4}(-0.1) - \frac{1}{64}(-0.1)^2 = \frac{12639}{6400}$$

$$= 1.97484375$$

6. State the Maclaurin series (using Σ-notation) for $\frac{1}{1-x}$ and then use it to find the Maclaurin series for $f(x) = \ln(1-x)$. You do not need to find its interval of convergence.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

$$f(x) = \ln(1-x) = \int \frac{-1}{1-x}\,dx = \int \sum_{n=0}^{\infty} (-1).x^n\,dx = C + \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1}$$

$$\left(x = 0 \Rightarrow \ln 1 = C + 0 \Rightarrow C = 0 \right)$$

[3]

$$\therefore \ln(1-x) = \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1} \quad \text{OR} \quad \sum_{n=1}^{\infty} \frac{-1}{n} x^n$$

7.

(a) Using known power series and trigonometric identities, find a Taylor series centred at $c = 0$ for the function $f(x) = x^3(2\cos^2 x - 1)$. Express your answer using Σ-notation.

$$f(x) = x^3 \cos 2x = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n} x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{2n+3}$$

[3]

(b) What is the interval of convergence of the Taylor series in part (a)? No work or justification is required.

[1] $(-\infty, \infty)$

8. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n3^n}$. Be sure to check the endpoints. Identify which tests you are using and show that all of the conditions of the tests are satisfied.

Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1)3^{n+1}}}{\frac{(x-2)^n}{n3^n}} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)n}{(n+1)3} \right|$$

$$= \frac{|x-2|}{3} \cdot \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x-2|}{3} \cdot \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{|x-2|}{3} \cdot 1 = \frac{|x-2|}{3}$$

$\therefore$ Absolute Convergence when $\frac{|x-2|}{3} < 1$

$\Rightarrow |x-2| < 3 \Rightarrow -3 < x-2 < 3 \Rightarrow -1 < x < 5$

[5]

Endpoints:

Check $x = -1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

$\therefore$ diverges (harmonic p-series $p=1$)

Check $x = 5$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

alt. series ✓

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\frac{1}{n+1} \le \frac{1}{n}$ for $n \ge 1$ ✓

$\therefore$ Converges by AST (conditionally)

$\therefore$ Interval of Convergence is $(-1, 5]$.