CAMOSUN

Name: SOLUTIONS

Mark: $\overline{}$ **25**

MATH 101 (Winter, 2023)

Test 3B

1. (5 marks) Consider the series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n+2}\right).$$

(a) Can the n^{th} Term Test be used to determine whether the series converges or diverges? Briefly explain.

(b) Find a formula for the n^{th} partial sum, S_n, of the series.
Hint: Write the series in the form of a telescoping series.

(c) Use your partial sum from part (b) to determine whether the series converges or diverges. If the series converges, find its sum. If it diverges, does it diverge to ∞, $-\infty$, or neither?

a) $\lim_{n\to\infty} \ln\left(\frac{n+1}{n+2}\right) = \lim_{n\to\infty} \ln\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right) = \ln 1 = 0$

$\therefore$ No. n^{th} Term Test cannot be used. It can only be used (to prove divergence) if $\lim_{n\to\infty} a_n \neq 0$.

b) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n+2}\right) = \sum_{n=1}^{\infty}\left(\underbrace{\ln(n+1)}_{b_n} - \underbrace{\ln(n+2)}_{b_{n+1}}\right)$ telescoping series!

$\therefore S_n = b_1 - b_{n+1} = \ln 2 - \ln(n+2)$

c) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n+2}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\ln 2 - \ln(n+2)\right) = -\infty$

$\therefore$ Series diverges to $-\infty$

2. (6 marks) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4n + 20}.$$

(a) Find the second partial sum, S_2, of the series.

(b) Verify that the conditions of the Integral Test are satisfied by the series.

(c) Using the Integral Test remainder formula, estimate the maximum error if the second partial sum were used to approximate the sum of the series.

a) $S_2 = \frac{1}{25} + \frac{1}{32} = \frac{57}{800}$

b) $f(x) = \frac{1}{x^2+4x+20}$ is clearly positive, continuous and decreasing for $x \geq 1$.

c) $0 < R_2 < \int_2^\infty \frac{1}{x^2+4x+20}\,dx$

$= \lim_{b\to\infty} \int_2^b \frac{1}{(x+2)^2+16}\,dx = \lim_{b\to\infty} \frac{1}{4}\arctan \frac{x+2}{4}\Big|_2^b$

$= \lim_{b\to\infty} \frac{1}{4}\left(\arctan \frac{b+2}{4} - \arctan 1\right)$

$= \frac{1}{4}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{16}$ ← Max error.

3. (6 marks) Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n}(x-3)^n.$$

Be sure to check endpoints, if applicable. Name any tests you use and check that all of their conditions are satisfied.

$$\lim_{n\to\infty}\left|\frac{\frac{(x-3)^{n+1}}{(n+1)4^{n+1}}}{\frac{(x-3)^n}{n4^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+1)4^{n+1}}\cdot\frac{n4^n}{(x-3)^n}\right| = \lim_{n\to\infty}\frac{n}{4(n+1)}|x-3|$$

$$= \lim_{n\to\infty}\frac{1}{4(1+\frac{1}{n})}|x-3| = \frac{1}{4}|x-3| < 1$$

absolute convergence by Ratio Test

$\Rightarrow |x-3| < 4$ radius of convergence is $R=4$

$\Rightarrow -4 < x-3 < 4$

$\Rightarrow -1 < x < 7$

Endpoints

$\underline{x=-1}$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n4^n}(-4)^n = -\underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}$$

diverges, harmonic p-series ($p=1$)

$\underline{x=7}$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n4^n}(4)^n = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

alt. series converges by AST since $\lim_{n\to\infty}\frac{1}{n}=0$ and $\frac{1}{n+1}\le\frac{1}{n}$ for $n\ge 1$.

$\therefore$ Interval of convergence is $(-1,7]$

4. (8 marks) Let $f(x) = \cosh x$.
   (a) Use the **definition** of a Maclaurin series to find the Maclaurin series for $f(x)$. Express your answer using Σ-notation.
   (b) Use your answer from part (a) to find a Maclaurin series for $\int x^4 \cosh(x^3)\, dx$. Express your answer using Σ-notation.
   (c) Find and simplify the 4^{th} Maclaurin polynomial, $P_4(x)$, for $f(x)$.
   (d) Use your answer from part (c) to approximate $\cosh(1)$.

a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cosh x$	1
1	$\sinh x$	0
2	$\cosh x$	1
3	$\sinh x$	0
4	$\cosh x$	1
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(0) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore f(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$

b)
$$\int x^4 \cosh(x^3)\,dx = \int x^4 \sum_{n=0}^{\infty} \frac{1}{(2n)!}(x^3)^{2n}\,dx$$
$$= \int x^4 \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{6n}\,dx = \int \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{6n+4}\,dx$$
$$= C + \sum_{n=0}^{\infty} \frac{1}{(2n)!\,(6n+5)} x^{6n+5}$$

c) $P_4(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$

d) $\cosh 1 \approx P_4(1) = 1 + \frac{1}{2} + \frac{1}{24} = \frac{37}{24}$