CAMOSUN

Name: SOLUTIONS

Mark: $\overline{}$ **25**

MATH 101 (Winter, 2023)

Test 2B

1. (4 marks) Evaluate the limit $\lim_{x\to1} \dfrac{2x-2-\ln x^2}{1+\cos \pi x}$. Form $\frac{0}{0}$

$$\overset{L'HR}{=} \lim_{x\to1} \frac{2-\frac{2}{x}}{-\pi \sin \pi x} \quad \text{Form } \frac{0}{0}$$

$$\overset{L'HR}{=} \lim_{x\to1} \frac{\frac{2}{x^2}}{-\pi^2 \cos \pi x} = \frac{2}{\pi^2}$$

2. (4 marks) Write the improper integral in the form of a limit and then evaluate it, or if it diverges, then determine whether it diverges to ∞, $-\infty$, or neither.

$$\int_1^\infty \frac{e^{1/x}}{x^2}\,dx$$

$$= \lim_{b\to\infty} -\int_1^b e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)dx = \lim_{b\to\infty} -e^{\frac{1}{x}}\Big|_1^b$$

$$= \lim_{b\to\infty}\left(-e^{\frac{1}{b}}+e\right) = -1+e \quad \text{(converges)}$$

3. Find the following integrals.

(a) (3 marks) $\int x^5 \ln x \, dx$

Let $u = \ln x$ $\quad dv = x^5 dx$

$du = \frac{1}{x} dx$ $\quad v = \frac{1}{6}x^6$

$\therefore \int x^5 \ln x \, dx = (\ln x)(\frac{1}{6}x^6) - \int (\frac{1}{6}x^6)(\frac{1}{x}) dx$

$= \frac{1}{6}x^6 \ln x - \frac{1}{6}\int x^5 dx$

$= \frac{1}{6}x^6 \ln x - \frac{1}{36}x^6 + C$

(b) (4 marks) $\int \frac{x^2 - 12}{x(x^2+4)} \, dx$

$\frac{x^2-12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \Rightarrow x^2-12 = A(x^2+4) + (Bx+c)x$

$= Ax^2 + 4A + Bx^2 + Cx$

$= (A+B)x^2 + Cx + 4A$

$4A = -12 \Rightarrow A = -3$

$C = 0$

$A+B = 1 \Rightarrow B = 4$

$\int \left(\frac{-3}{x} + \frac{4x}{x^2+4}\right) dx = -3\ln|x| + 2\ln(x^2+4) + C$

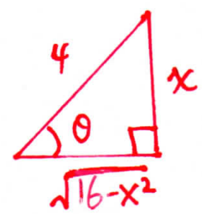
4. (5 marks) Use a trigonometric substitution to evaluate the integral $\int_0^2 \frac{x^2}{\sqrt{16-x^2}}\,dx$.

Let $x = 4\sin\theta$

$\sin\theta = \frac{x}{4}$

$\theta = \arcsin\frac{x}{4}$

$dx = 4\cos\theta\, d\theta$



$\sqrt{16-x^2} = 4\cos\theta$

$x = 2 \Rightarrow \theta = \arcsin\frac{1}{2} = \frac{\pi}{6}$

$x = 0 \Rightarrow \theta = \arcsin 0 = 0$

$$= \int_0^{\frac{\pi}{6}} \frac{(4\sin\theta)^2}{4\cos\theta} \cdot 4\cos\theta\, d\theta$$

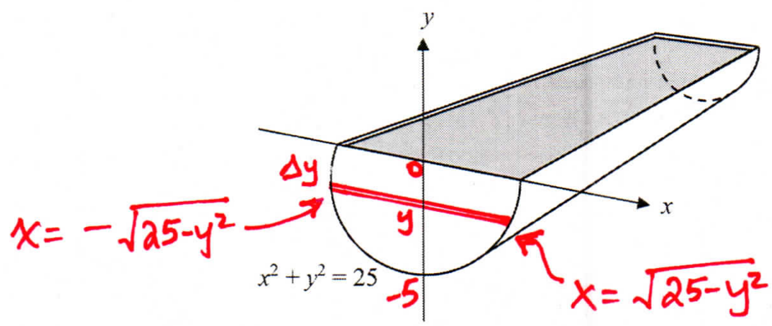
$$= 16\int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta$$

$$= 16\int_0^{\frac{\pi}{6}} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \Big[8\theta - 4\sin 2\theta\Big]_0^{\pi/6}$$

$$= \frac{4\pi}{3} - 2\sqrt{3}$$

5. (5 marks) Each end of a swimming pool is in the shape of the semicircle $x^2 + y^2 = 25$ for $y \leq 0$, where x and y are measured in feet, as illustrated. If the pool is full of water weighing $\underbrace{62.4 \text{ lb/ft}^3}_{w}$, then how much force does the water exert on the wall at each end of the pool?



$$\Delta F = P\Delta A = w\,h(y)\,L(y)\,\Delta y \quad \text{for } -5 \leq y \leq 0$$

where $h(y) = 0 - y = -y$

and $L(y) = \sqrt{25-y^2} - \left(-\sqrt{25-y^2}\right) = 2\sqrt{25-y^2}$

$$\therefore\ F = w\int_{-5}^{0} (-y)\,2\sqrt{25-y^2}\,dy = w\int_{-5}^{0} (25-y^2)^{1/2}(-2y)\,dy$$

$$= \frac{2}{3}w(25-y^2)^{3/2}\Big|_{-5}^{0} = \frac{2}{3}w\left(125-0\right)$$

$$= \frac{250}{3}(62.4) = 5200 \text{ lb}$$