Integral Test

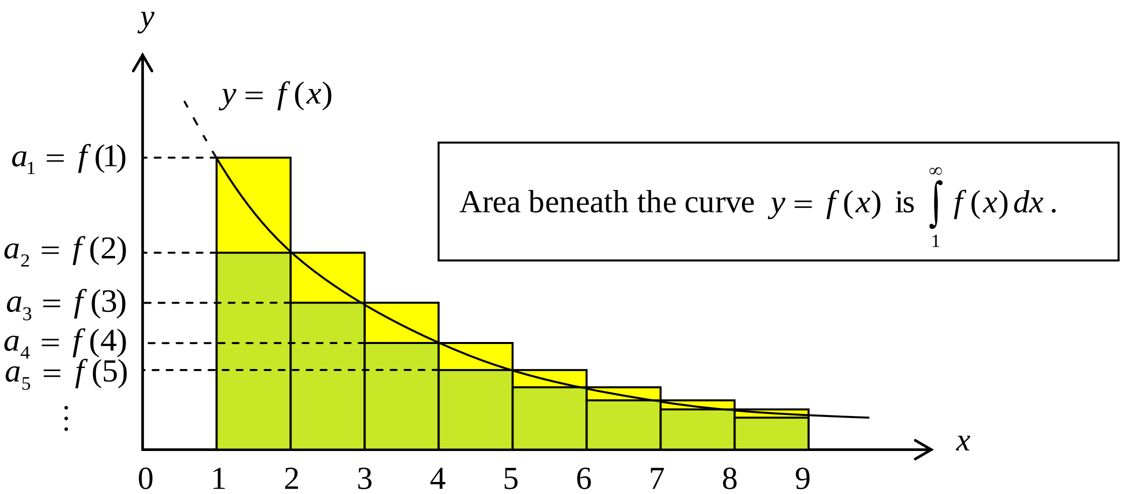
Integral Test: If f is positive, continuous and decreasing for $x \geq N$, where N is some fixed positive integer, and $a_n = f(n)$ for $n \geq N$, then

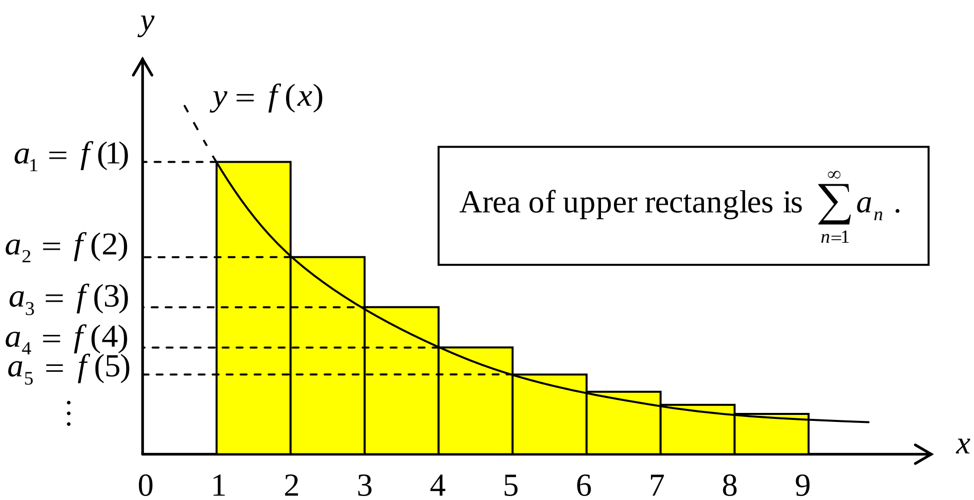
$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \int_N^{\infty} f(x)\,dx$$

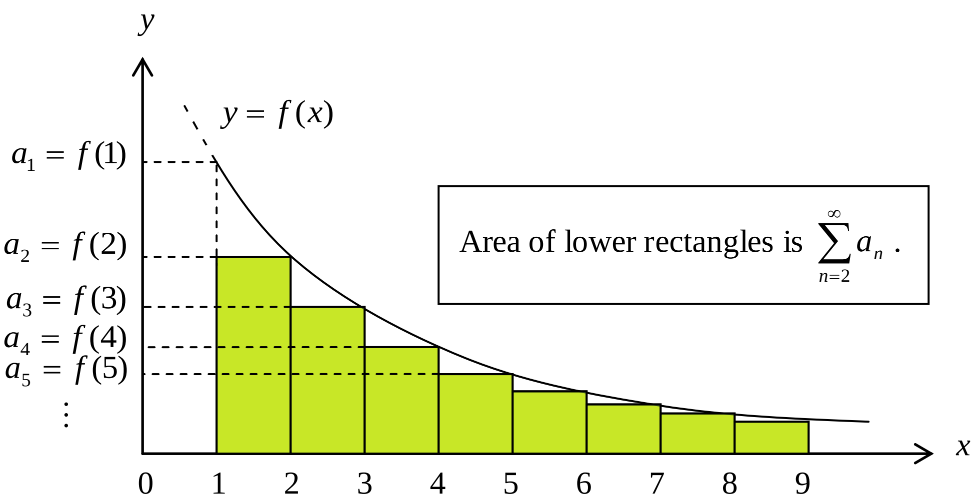
either both converge or they both diverge to infinity.

Note: In examples, N is often equal to 1. Moreover, in the case of convergence the series and the improper integral **do not** converge to the same value. In other words,

$$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx .$$







$$\boxed{\sum_{n=2}^{\infty} a_n \le \int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} a_n}$$

Let $\sum_{n=1}^{\infty} a_n$ be a series that converges according to the Integral Test. Let

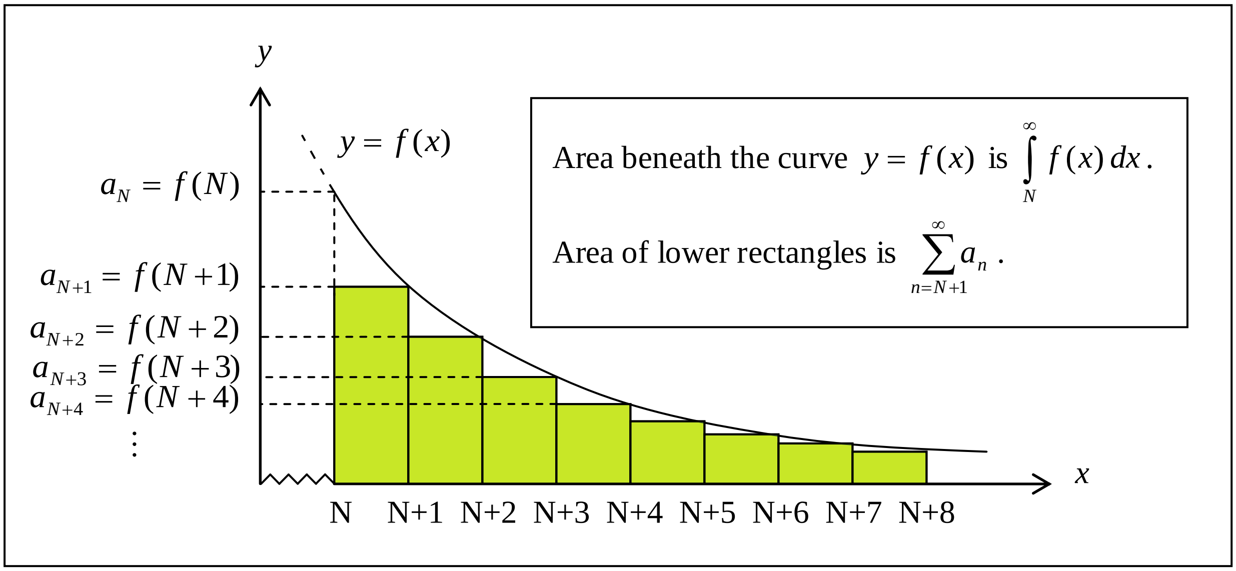
$$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$

be its sum. The N^{th} partial sum

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_N$$

can be used to approximate S. In other words, $S \approx S_N$. The error associated with this approximation is called the **remainder** and it is given by

$$R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n = a_{N+1} + a_{N+2} + a_{N+3} + a_{N+4} + a_{N+5} + \cdots$$



The remainder R_N is equal to the area of the lower rectangles. Therefore,

$$0 < R_N < \int_N^{\infty} f(x)\,dx.$$

The integral $\int_N^{\infty} f(x)\,dx$ represents the maximum error in approximating S by S_N. Finally, since $S = S_N + R_N$, then the exact sum S must belong to the interval

$$S_N < S < S_N + \int_N^{\infty} f(x)\,dx.$$