Integral Test

Integral Test: If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$ for $n \geq 1$, then

$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \int_1^{\infty} f(x)\,dx$$

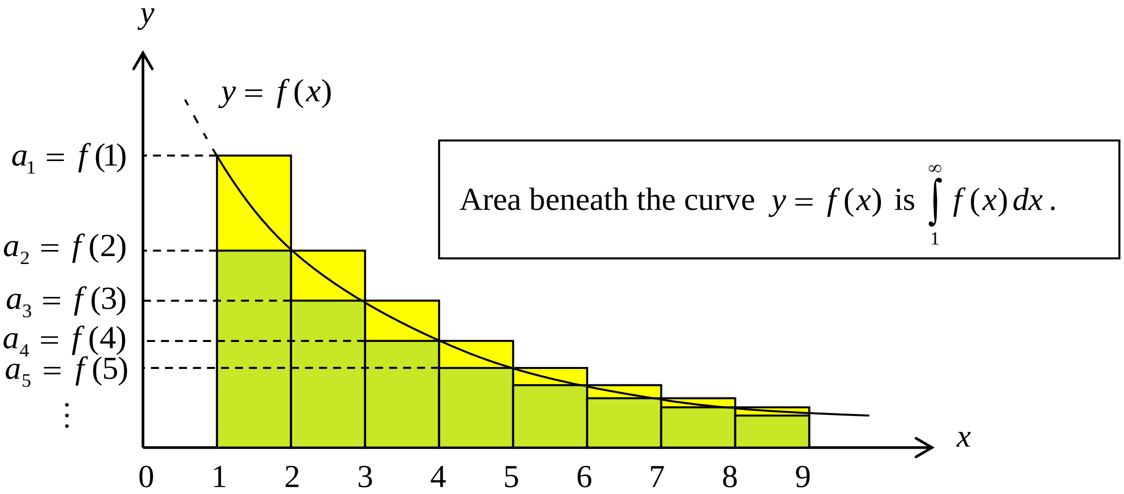
either both converge or they both diverge to infinity.

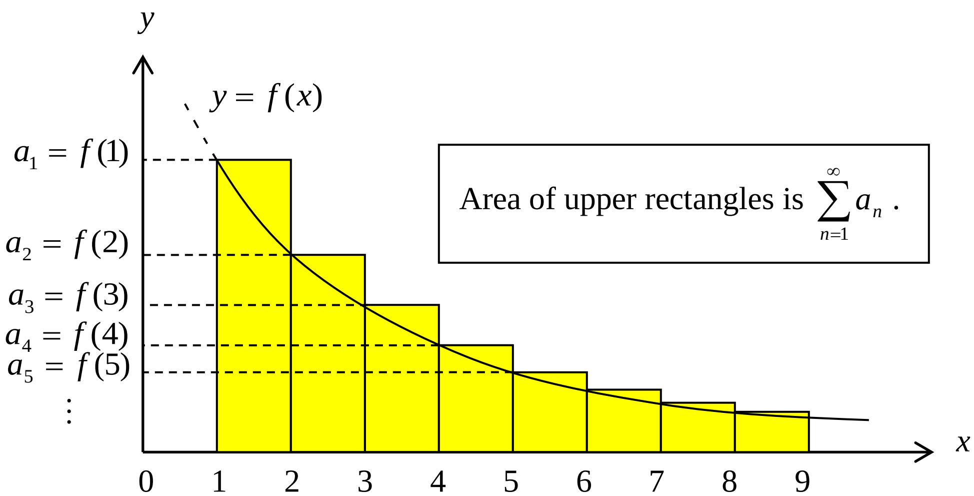
Note: Since the convergence or divergence behaviour of the series is not affected by the values at the beginning of the series, the conditions of the theorem can be weakened somewhat by only requiring that $x \geq N$ and $n \geq N$, where N is some fixed positive integer, not necessarily 1. In this case you would test for convergence the integral

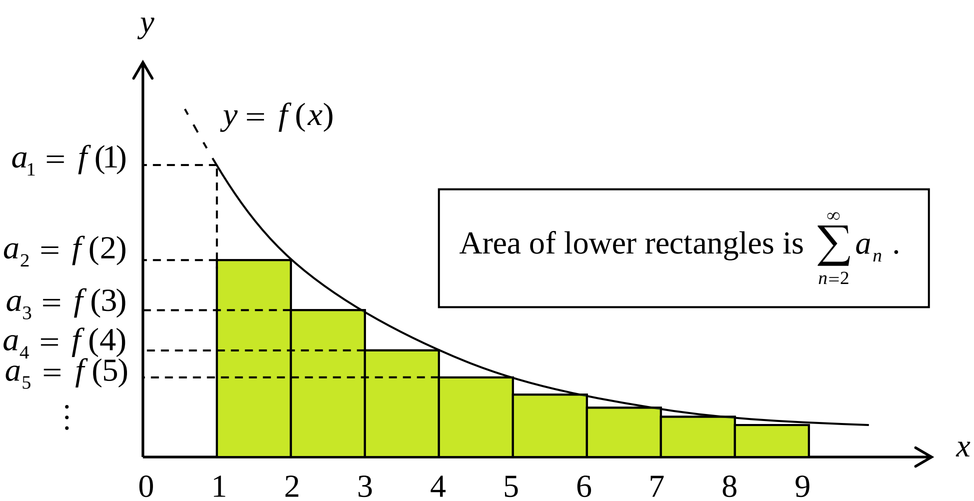
$$\int_N^{\infty} f(x)\,dx .$$

Note: In general, the sum of the infinite series and the value of the improper integral are **not** equal.

Integral Test Illustration







$$\boxed{\sum_{n=2}^{\infty} a_n \le \int_1^\infty f(x)\,dx \le \sum_{n=1}^{\infty} a_n}$$

Integral Test Remainder

Let $\sum_{n=1}^{\infty} a_n$ be a series that converges according to the Integral Test. Let

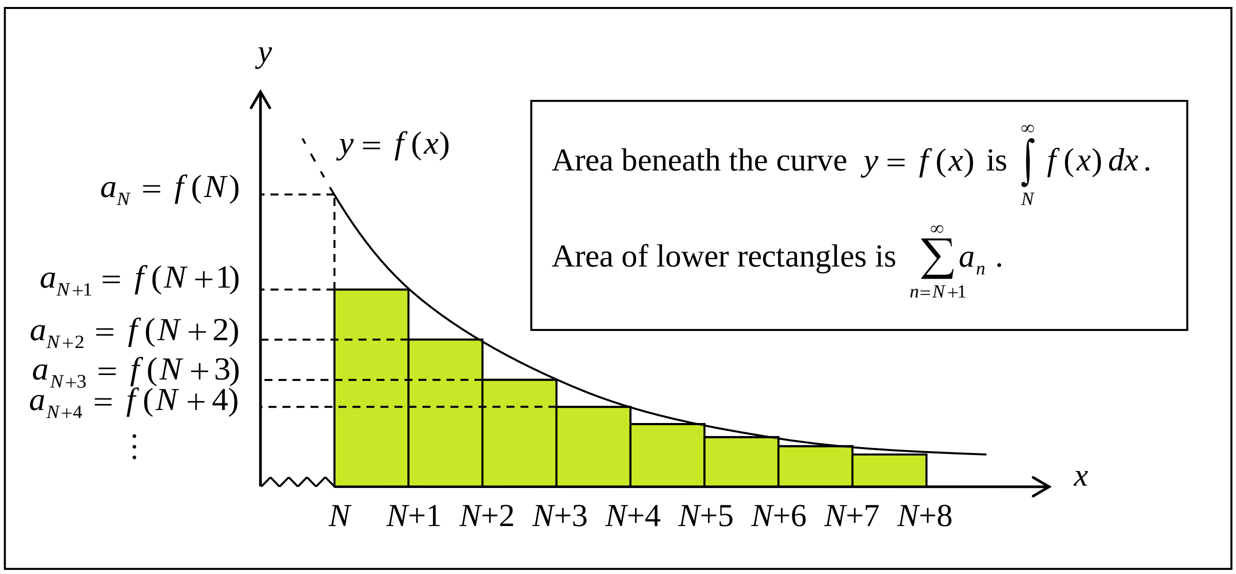
$$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$

be its sum. The N^{th} partial sum

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_N$$

can be used to approximate *S*. In other words, $S \approx S_N$. The error associated with this approximation is called the **remainder** and it is given by

$$R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n = a_{N+1} + a_{N+2} + a_{N+3} + a_{N+4} + a_{N+5} + \cdots$$



The remainder R_N is equal to the area of the lower rectangles. Therefore,

$$0 < R_N < \int_N^{\infty} f(x)\,dx.$$

The integral $\int_N^{\infty} f(x)\,dx$ represents the maximum error in approximating *S* by S_N. Finally, since $S = S_N + R_N$, then the exact sum *S* must belong to the interval

$$S_N < S < S_N + \int_N^{\infty} f(x)\,dx.$$