Examples of Absolute vs. Conditional Convergence

The following series all converge according to the Alternating Series Test:

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{2^n}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{e^n}.$$

All three series are alternating series; note that $\cos(n\pi) = (-1)^n$.

Secondly, $\lim_{n\to\infty} a_n = 0$ in each case. In other words,

$$\lim_{n\to\infty} \frac{1}{2^n} = 0, \qquad \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0, \qquad \lim_{n\to\infty} \frac{n}{e^n} = 0.$$

These limits are obvious in the first two cases. To prove the third case we convert to x and use L'Hôpital's Rule,

$$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

Thirdly, $a_{n+1} \leq a_n$ for $n \geq 1$ in all three series. In other words,

$$\frac{1}{2^{n+1}} \leq \frac{1}{2^n}, \qquad \frac{1}{\ln(n+2)} \leq \frac{1}{\ln(n+1)}, \qquad \frac{n+1}{e^{n+1}} \leq \frac{n}{e^n}.$$

Again this is obvious in the first two cases but not the third. This is because the numerator $n+1$ is actually larger than the numerator n. Nevertheless, the third case can be proven algebraically by a clever use of inequalities (or by using derivatives from calculus as we describe later). Observe that $e - 1 \approx 1.718 \geq 1$ and so $(e-1)n \geq 1$ for $n \geq 1$. Therefore

$$\frac{n+1}{e^{n+1}} \leq \frac{n + (e-1)n}{e^{n+1}} = \frac{en}{e^{n+1}} = \frac{n}{e^n}.$$

Since these three series converge, we now want to distinguish between absolute and conditional convergence by considering the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}, \qquad \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}, \qquad \sum_{n=1}^{\infty} \frac{n}{e^n}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

is a geometric series with $r = 1/2$, which converges since $|r| < 1$.

We therefore conclude that the series

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{2^n}$$

converges absolutely.

The series

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$$

diverges since it can be compared to the divergent harmonic series by using the Direct Comparison Test (the Limit Comparison Test would prove to be inconclusive). The claim is that

$$\frac{1}{\ln(n+1)} \geq \frac{1}{n},$$

for $n \geq 1$. This inequality is equivalent to

$$n \geq \ln(n+1),$$

which in turn is equivalent to

$$n - \ln(n+1) \geq 0.$$

This statement is true, but it requires justification since it's not obvious. We can prove this inequality by introducing a function of a real variable x given by $f(x) = x - \ln(x+1)$ for $x \geq 0$ and noting that $f(0) = 0$. Differentiating gives us

$$f'(x) = 1 - \frac{1}{x+1} = \frac{x}{x+1} \geq 0,$$

for $x \geq 0$. Therefore f is an increasing function on the interval $x \geq 0$. Together with the fact that $f(0) = 0$, this proves $f(x) \geq 0$ for all $x \geq 0$. Since $x - \ln(x+1) \geq 0$ for all $x \geq 0$, then in particular for positive integer values of n we get $n - \ln(n+1) \geq 0$.

We therefore conclude that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)},$$

converges conditionally.

The series

$$\sum_{n=1}^{\infty} \frac{n}{e^n},$$

can be shown to converge by using the Integral Test. Let

$$f(x) = \frac{x}{e^x} = xe^{-x},$$

for $x \geq 1$. This function is clearly continuous and positive for $x \geq 1$. However, it's not obvious that it's decreasing. We can verify that it's decreasing by showing that its derivative is nonpositive. Differentiating gives us

$$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x) \leq 0,$$

for $x \geq 1$. This can also be used as an alternative way of justifying why $a_{n+1} \leq a_n$ for all $n \geq 1$ when applying the Alternating Series Test.

Applying the Integral Test involves evaluating the improper integral (by parts)

$$\begin{aligned}
\int_1^{\infty} xe^{-x}\, dx &= \lim_{b \to \infty} \int_1^b xe^{-x}\, dx = \lim_{b \to \infty} \left[-xe^{-x} - e^{-x}\right]_1^b \\
&= \lim_{b \to \infty} \left[-be^{-b} - e^{-b} + 2e^{-1}\right] = 2e^{-1} - \lim_{b \to \infty} \frac{b+1}{e^b} \\
&= 2e^{-1} - \lim_{b \to \infty} \frac{1}{e^b} = 2e^{-1},
\end{aligned}$$

where L'Hôpital's Rule was used to evaluate the limit. Since the integral converges, then the series

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

converges.

We therefore conclude that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{e^n}$$

converges absolutely.

We note that in cases like this where we are proving absolute convergence, strictly speaking it is unnecessary to first prove ordinary convergence as we did using the Alternating Series Test.

Finally, we note that a far easier way of proving that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{e^n}$$

converges absolutely would have been to use a new (and arguably the most important) test for convergence/divergence of series known as the Ratio Test.