Alternating Series

An **alternating series** is a series whose terms alternate in sign, having the form

$$\sum_{n=1}^{\infty}(-1)^n a_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

where $a_n > 0$ represents the absolute value of the n^{th} term of the series. Two examples are

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \qquad \text{(the alternating harmonic series)}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(2n+1)^2} = -\frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \frac{1}{121} + \cdots$$

Other ways of writing the alternating signs include $(-1)^{n-1} = (-1)^{n+1}$ and $\cos n\pi = (-1)^n$.

The following test guarantees convergence of an alternating series.

Alternating Series Test (AST): An alternating series converges if both of the following conditions are satisfied.

1. $\lim_{n\to\infty} a_n = 0$, and
2. $a_{n+1} \leq a_n$ for all $n \geq N$, where N is some fixed positive integer

Note that if condition 1 is not satisfied, then the series will diverge by the n^{th} Term Test.

The following theorem provides an upper bound on the error associated with using partial sums to approximate the sum of a convergent alternating series.

Alternating Series Remainder (ASR) Theorem: Suppose an alternating series satisfies the conditions of the AST with $N = 1$. Let S be the sum of the series and let S_N be the N^{th} partial sum. If $R_N = S - S_N$ is the remainder (or error), then

$$|R_N| \leq a_{N+1}.$$

Moreover, if a_{N+1} is being added within the series, then

$$0 \leq R_N \leq a_{N+1} \qquad \text{and} \qquad S_N \leq S \leq S_N + a_{N+1},$$

while if a_{N+1} is being subtracted within the series, then

$$-a_{N+1} \leq R_N \leq 0 \qquad \text{and} \qquad S_N - a_{N+1} \leq S \leq S_N.$$