CAMOSUN

Name: SOLUTIONS

Mark: $\overline{}$ **25**

MATH 101 (Fall, 2025)

Test 3

1. (2 marks) Answer whether the statements are True or False. No justification is required.

(a) F (b) F (c) T (d) T

(a) If the sequence $\{a_n\}$ converges to zero, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $0 < a_n \le b_n$ for all $n \ge 1$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

(c) If $a_n \neq 0$ for all $n \ge 1$ and $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

(d) If $a_n > 0$ and $b_n > 0$ for all $n \ge 1$, $\sum_{n=1}^{\infty} a_n$ converges, and $\lim_{n\to\infty} \frac{b_n}{a_n} = 2$, then $\sum_{n=1}^{\infty} b_n$ converges.

2. (3 marks) Determine whether the series converges conditionally, converges absolutely, or diverges. Identify which tests you are using and show that all of the conditions of the tests are satisfied.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$$

alt. series ✓ $\lim_{n\to\infty} \frac{1}{\sqrt[4]{n}} = 0$ ✓ $\frac{1}{\sqrt[4]{n+1}} \le \frac{1}{\sqrt[4]{n}}$ for $n \ge 1$ ✓

$\therefore$ series converges by AST

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$ diverges by p-series test since $p = \frac{1}{4} < 1$.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges conditionally

3. Find the sum of each convergent series.

(a) (2 marks) $\sum_{n=1}^{\infty} \left(4^{1/n} - 4^{1/(n+1)}\right)$ $= 4^{\frac{1}{1}} - \lim_{n\to\infty} 4^{\frac{1}{n+1}} = 4 - 4^0 = 4 - 1 = 3$

telescoping series

(b) (1 mark) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\pi^{2n}$ $= \cos\pi = -1$

4. (3 marks) Determine whether the series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$ converges or diverges. Identify which test you are using.

Ratio Test $\lim_{n\to\infty} \dfrac{\frac{[2(n+1)]!}{[(n+1)!]^2}}{\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty} \frac{(2n+2)!}{[(n+1)!]^2} \cdot \frac{(n!)^2}{(2n)!}$

$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)!}}{[(n+1)\cancel{n!}]^2} \cdot \frac{\cancel{(n!)^2}}{\cancel{(2n)!}} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2}$

$= \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1} = \lim_{n\to\infty} \frac{4+\frac{6}{n}+\frac{2}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}} = \frac{4}{1} = 4 > 1$

$\therefore$ series diverges

or $\frac{4n+2}{n+1}$ if reduced by cancelling factors of $n+1$

5. Let $f(x) = \cosh x$.

(a) (2 marks) Find and simplify the 4^{th} Maclaurin polynomial, $P_4(x)$, for $f(x)$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cosh x$	1
1	$\sinh x$	0
2	$\cosh x$	1
3	$\sinh x$	0
4	$\cosh x$	1

$$P_4(x) = 1 + 0x + \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4$$
$$= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$$

(b) (1 mark) Use $P_4(x)$ from part (a) to approximate $\cosh(1/2)$. Express your answer as a fraction reduced to lowest terms.

$$\cosh\tfrac{1}{2} \approx P_4\left(\tfrac{1}{2}\right) = 1 + \tfrac{1}{2}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{24}\left(\tfrac{1}{2}\right)^4 = 1 + \tfrac{1}{8} + \tfrac{1}{384} = \frac{433}{384}$$

(c) (1 mark) Use the **definition** of Maclaurin series to find the Maclaurin series for $f(x)$. Express your answer using Σ-notation.

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$

6. (3 marks) Use a power series for $\sin x$ to find a Taylor series for $f(x) = 5x\sin(5x^2)$ centered at $c = 0$. Express your answer using Σ-notation. What is the interval of convergence of the Taylor series?

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ for } -\infty < x < \infty$$

$$\therefore f(x) = 5x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (5x^2)^{2n+1} = 5x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 5^{2n+1} x^{4n+2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 5^{2n+2} x^{4n+3}$$

Interval of convergence is $(-\infty, \infty)$.

7. Let $f(x) = \sum_{n=0}^{\infty} \frac{3}{(n+1)5^n}(x+1)^{n+1}$ be a power series function centered at $c = -1$.

(a) (1 mark) Find $f'(x)$, expressed in the form of a power series centered at $c = -1$.

$$f'(x) = \sum_{n=0}^{\infty} \frac{3}{(n+1)5^n} \cdot (n+1)(x+1)^n = \sum_{n=0}^{\infty} \frac{3}{5^n}(x+1)^n \text{ or } \sum_{n=0}^{\infty} 3\left(\frac{x+1}{5}\right)^n$$

(b) (1 mark) What type of series is the series in part (a)?

geometric power series

(c) (2 marks) Find the interval of convergence of the power series for $f'(x)$ found in part (a).

Converges iff $|r|<1$ where $r = \frac{x+1}{5}$.

$$\left|\frac{x+1}{5}\right| < 1 \Rightarrow |x+1| < 5 \Rightarrow -5 < x+1 < 5 \Rightarrow -6 < x < 4$$

$\therefore$ Interval of convergence is $(-6, 4)$.

(Note: geometric power series diverges at endpoints. No need to test.)

(d) (2 marks) Evaluate $f'(1)$.

$$f'(1) = \sum_{n=0}^{\infty} 3\left(\frac{2}{5}\right)^n = \frac{3}{1-2/5} = \frac{3}{3/5} = 5$$

Note that 1 is in $(-6,4)$.

(e) (1 mark) Find $\int f(x)\,dx$, expressed in the form of a power series centered at $c = -1$.

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{3}{(n+2)(n+1)5^n}(x+1)^{n+2}$$