CAMOSUN

Name: SOLUTIONS

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MATH 101 (Fall, 2025)

Test 2

1. Find the following integrals.

   (a) (3 marks) $\int \sin^3 7\theta \, d\theta$

$$= \int \sin^2 7\theta \sin 7\theta \, d\theta = -\frac{1}{7}\int (1-\cos^2 7\theta)(-7\sin 7\theta)\, d\theta$$

$$= -\frac{1}{7}\left(\cos 7\theta - \frac{1}{3}\cos^3 7\theta\right) + C = -\frac{1}{7}\cos 7\theta + \frac{1}{21}\cos^3 7\theta + C$$

   (b) (3 marks) $\int \arctan x \, dx$

$$= x\arctan x - \int \frac{x}{1+x^2}\, dx$$

$$\begin{pmatrix} u = \arctan x & dv = dx \\ du = \frac{1}{1+x^2}\, dx & v = x \end{pmatrix}$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

2. (3 marks) Evaluate the improper integral, or if it diverges, then determine whether it diverges to ∞, $-\infty$ or neither. Be sure to convert the integral to a limit.

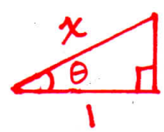
$$\int_e^\infty \frac{(\ln x)^3}{x}\,dx$$

$$= \lim_{b\to\infty}\int_e^b \frac{(\ln x)^3}{x}\,dx = \lim_{b\to\infty} \frac{1}{4}(\ln x)^4\Big|_e^b = \lim_{b\to\infty}\left(\frac{1}{4}(\ln b)^4 - \frac{1}{4}\right) = \infty$$

$\therefore$ diverges to ∞

3. (4 marks) Use a trigonometric substitution to integrate $\displaystyle\int \sqrt{x^2-1}\,dx$.

Let $x = \sec\theta$
$dx = \sec\theta\tan\theta\,d\theta$



$\sqrt{x^2-1} = \tan\theta$

$$\int\sqrt{x^2-1}\,dx = \int \tan\theta\cdot\sec\theta\tan\theta\,d\theta = \int\tan^2\theta\sec\theta\,d\theta$$

$$= \int(\sec^2\theta-1)\sec\theta\,d\theta = \int(\sec^3\theta-\sec\theta)\,d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta| + C$$

$$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln|x+\sqrt{x^2-1}| + C$$

4. (5 marks) Use partial fractions to integrate $\displaystyle\int \frac{5x^2+2x}{(x-2)(x^2+4)}\,dx$.

$$\frac{5x^2+2x}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4} \Rightarrow 5x^2+2x = A(x^2+4) + (Bx+c)(x-2)$$

$$x=2 \Rightarrow 24 = 8A \Rightarrow A=3$$

$$x=0 \Rightarrow 0 = 4A-2C \Rightarrow 0 = 12-2C \Rightarrow C=6$$

$$x=1 \Rightarrow 7 = 5A-B-C \Rightarrow 7 = 15-B-6 \Rightarrow B=2$$

$$\therefore \int\left(\frac{3}{x-2} + \frac{2x+6}{x^2+4}\right)dx = \int\left(\frac{3}{x-2} + \frac{2x}{x^2+4} + \frac{6}{x^2+4}\right)dx$$

$$= 3\ln|x-2| + \ln(x^2+4) + 3\arctan\frac{x}{2} + C$$

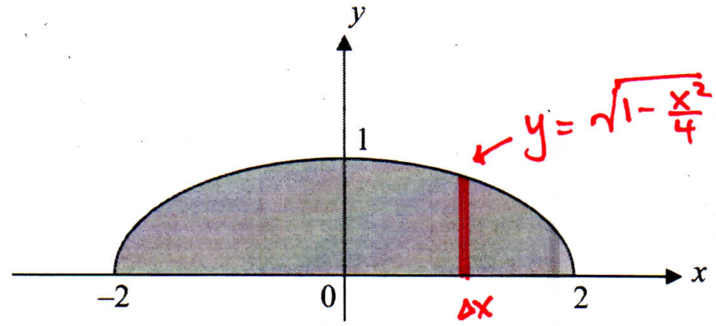
5. (3 marks) Evaluate the limit, or if it does not exist, then determine whether it is ∞, $-\infty$ or neither.

$$\lim_{x\to\infty} x\tan\frac{1}{x}$$ Form $\infty\cdot 0$

$$= \lim_{x\to\infty} \frac{\tan\frac{1}{x}}{\frac{1}{x}} \overset{L'HR}{=} \lim_{x\to\infty} \frac{\sec^2\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \sec^2\frac{1}{x} = 1$$

Form $\frac{0}{0}$

6. (4 marks) Find the coordinates of the centroid of the semielliptical region bounded by the curves $\frac{x^2}{4} + y^2 = 1$ and $y = 0$, for $y \geq 0$, given that the area of the region is $A = \pi$.



By symmetry, $A_y = 0 \Rightarrow \bar{x} = 0$.

For $-2 \leq x \leq 2$, $\Delta A_x = \frac{1}{2}\sqrt{1-\frac{x^2}{4}} \cdot \sqrt{1-\frac{x^2}{4}}\,\Delta x = \frac{1}{2}\left(1 - \frac{x^2}{4}\right)\Delta x$

$\therefore A_x = \frac{1}{2}\int_{-2}^{2}\left(1-\frac{x^2}{4}\right)dx \underset{\text{by symmetry}}{=} \int_0^2\left(1-\frac{x^2}{4}\right)dx = \left[x - \frac{x^3}{12}\right]_0^2 = 2 - \frac{2}{3} = \frac{4}{3}$

$\therefore \bar{y} = \frac{A_x}{A} = \frac{4/3}{\pi} = \frac{4}{3\pi}$

$\therefore (\bar{x}, \bar{y}) = \left(0, \frac{4}{3\pi}\right)$.