MATH 101
Sequences and Series
True or False?

(a) $\sum_{n=1}^{\infty} n^{-n}$ converges.

(b) If the sequence $\{a_n\}$ diverges and the sequence $\{b_n\}$ diverges, then the sequence $\{a_n + b_n\}$ diverges.

(c) If $\lim_{n\to\infty} a_n$ exists, then $\sum_{n=1}^{\infty} a_n$ converges.

(d) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

(e) If the sequence $\{a_n\}$ converges, then $\lim_{n\to\infty}(a_n - a_{n+1}) = 0$.

(f) If $a_n \neq 0$ for all $n \geq 1$ and $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(g) If $0 < a_n \leq b_n$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

(h) If $|r| < 1$, then $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$.

(i) If $f(x)$ is a positive, continuous and decreasing function for $x \geq 1$ and if $a_n = f(n)$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n = \int_1^{\infty} f(x)\,dx$.

(j) If both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} (-a_n)$ converge, then so does $\sum_{n=1}^{\infty} |a_n|$.

(k) If $a_n > 0$ and $b_n > 0$ for all $n \geq 1$, $\sum_{n=1}^{\infty} a_n$ converges and $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$, then $\sum_{n=1}^{\infty} b_n$ converges.

(l) $(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + (a_4 - a_5) + (a_5 - a_6) + (a_6 - a_7) + \cdots = a_1$

Solutions

(a) T	(b) F	(c) F	(d) T	(e) T	(f) F	(g) F	(h) T	(i) F	(j) F	(k) T	(l) F

True (a) Since $n^{-n} = \frac{1}{n^n} \le \frac{1}{n^2}$ for $n \ge 1$ and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by the Direct Comparison Test $\sum_{n=1}^{\infty} n^{-n}$ also converges. Alternatively, since $n^{-n} = \left(\frac{1}{n}\right)^n \le \left(\frac{1}{2}\right)^n$ for $n \ge 2$ and the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges, then by the Direct Comparison Test $\sum_{n=1}^{\infty} n^{-n}$ also converges. The Ratio Test (or Root Test) can also be used to show the series converges.

False (b) For a counterexample, let $a_n = n$ and $b_n = -n$ for all $n \ge 1$. Then $\lim_{n\to\infty} a_n = \infty$ (diverges) and $\lim_{n\to\infty} b_n = -\infty$ (diverges); however, $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} 0 = 0$ (converges).

False (c) For a counterexample, let $a_n = 1$ for all $n \ge 1$. Then $\lim_{n\to\infty} a_n = 1$ (converges) while $\sum_{n=1}^{\infty} a_n = \infty$ (diverges). Note that even if $\lim_{n\to\infty} a_n$ exists and equals 0, the series $\sum_{n=1}^{\infty} a_n$ may still diverge as in the harmonic series, where $a_n = \frac{1}{n}$ for $n \ge 1$.

True (d) This statement is logically equivalent to the contrapositive statement "if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges," which we know to be true by Theorem 9.16. Recall that if $\sum_{n=1}^{\infty} |a_n|$ converges, then we say $\sum_{n=1}^{\infty} a_n$ converges absolutely.

True (e) Since $\{a_n\}$ converges, let $L = \lim_{n\to\infty} a_n$. Then it's also true that $\lim_{n\to\infty} a_{n+1} = L$. By properties of limits (Theorem 1.2), $\lim_{n\to\infty}(a_n - a_{n+1}) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} a_{n+1} = L - L = 0$.

False (f) The limit inequality $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| < 1$ is equivalent to saying $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$.

Therefore, the conclusion according to the Ratio Test should be that the series $\sum_{n=1}^{\infty} a_n$ diverges.

False (g) Knowing that $\sum_{n=1}^{\infty} a_n$ converges says nothing about the convergence of $\sum_{n=1}^{\infty} b_n$. For instance, if $a_n = b_n = \frac{1}{2^n}$ for $n \ge 1$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge to 1. On the other hand if $a_n = \frac{1}{2^n}$

but $b_n = 1$ (so that $0 < a_n \le b_n$ for all n), then $\sum_{n=1}^{\infty} a_n$ converges, however $\sum_{n=1}^{\infty} b_n$ diverges. Note that if $\sum_{n=1}^{\infty} a_n$ had diverged, then $\sum_{n=1}^{\infty} b_n$ would have diverged also by the Direct Comparison Test.

<u>True</u> (h) For $|r| < 1$, $\sum_{n=1}^{\infty} r^n$ is just the geometric series $\sum_{n=0}^{\infty} r^n$, which converges to $\dfrac{1}{1-r}$, except it is missing the first term $r^0 = 1$ when $n = 0$. Therefore $\sum_{n=1}^{\infty} r^n = \dfrac{1}{1-r} - 1 = \dfrac{r}{1-r}$. To see this another way, write out the terms of the series and factor out an r:
$\sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + r^4 + \cdots = r(1 + r + r^2 + r^3 + \cdots) = r\sum_{n=0}^{\infty} r^n = r \cdot \dfrac{1}{1-r} = \dfrac{r}{1-r}$. For a third approach notice that the first term of the geometric series $r + r^2 + r^3 + r^4 + \cdots$ is $a = r$ and its common ratio is r and so the sum is $\dfrac{a}{1-r} = \dfrac{r}{1-r}$.

<u>False</u> (i) According to the Integral Test, the conclusion should be that the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge. If they both converge, they will not converge to the same number. In fact the integral $\int_1^{\infty} f(x)\,dx$ will be *less than* $\sum_{n=1}^{\infty} a_n$ (but be greater than $\sum_{n=2}^{\infty} a_n$). For a counterexample, let $f(x) = e^{-x}$. Then $\sum_{n=1}^{\infty} e^{-n} = \dfrac{1}{e-1} \approx 0.582$ while $\int_1^{\infty} e^{-x}\,dx = \dfrac{1}{e} \approx 0.368$.

<u>False</u> (j) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-a_n) = -\sum_{n=1}^{\infty} a_n$ must also converge, but this does not imply $\sum_{n=1}^{\infty} |a_n|$ converges. It is possible for $\sum_{n=1}^{\infty} |a_n|$ to diverge (in which case we would say that $\sum_{n=1}^{\infty} a_n$ converges conditionally). For a counterexample, let $a_n = \dfrac{(-1)^n}{n}$ for $n \ge 1$. Then both $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ and $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ converge; however $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

<u>True</u> (k) This is a direct consequence of the Limit Comparison Test where $L = 1$ is finite and positive.

<u>False</u> (l) The sum of this telescoping series is $a_1 - \lim_{n\to\infty} a_{n+1}$ if the limit exists, not just a_1. For example, if $a_n = 1$ for $n \ge 1$, then the sum of the series is 0, not $a_1 = 1$. Alternatively, if $a_n = n$ for $n \ge 1$, then the series diverges to $-\infty$ and does not equal $a_1 = 1$. Only if $\lim_{n\to\infty} a_{n+1} = 0$ (or equivalently $\lim_{n\to\infty} a_n = 0$) will the series converge to a_1.