

Derivation of Euler's Formula and Euler's Identity using Taylor Series

Since

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

then

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{1}{n!} (ix)^n = \sum_{n=0}^{\infty} \frac{i^n}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{(2n+1)!} x^{2n+1} && \text{splitting into even and odd powers} \\ &= \sum_{n=0}^{\infty} \frac{(i^2)^n}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(i^2)^n}{(2n+1)!} x^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \cos x + i \sin x \end{aligned}$$

Thus

$$e^{ix} = \cos x + i \sin x$$

Euler's Formula

Letting $x = \pi$ gives

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$$

which upon rearranging becomes

$$e^{i\pi} + 1 = 0$$

Euler's Identity



Arguably the most famous of all mathematics equations, Euler's Identity is elegant in its simplicity. It's a valid formula that incorporates the five most important numbers in mathematics: the additive and multiplicative identities 0 and 1, the number i , which is the building block of the complex numbers, and two of the most important irrational numbers in mathematics, π and e . It also involves the three most important mathematical operations: addition, multiplication and exponentiation.