Derivation of Euler's Formula and Euler's Identity using Taylor Series

Since

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

then

$$e^{ix} = \sum_{n=0}^{\infty} \frac{1}{n!} (ix)^n = \sum_{n=0}^{\infty} \frac{i^n}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{(2n+1)!} x^{2n+1} \qquad \text{splitting into even and odd powers}$$

$$= \sum_{n=0}^{\infty} \frac{(i^2)^n}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(i^2)^n}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \cos x + i \sin x$$

Thus

$$\boxed{e^{ix} = \cos x + i \sin x}$$

Letting $x = \pi$ gives

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$$

which upon rearranging becomes

$$\boxed{e^{i\pi} + 1 = 0}$$