CAMOSUN

Name: SOLUTIONS

Mark: $\frac{\quad}{25}$

MATH 101
Assignment 7

1. (2 marks) Classify the graph of the following equation as a circle, parabola, ellipse or hyperbola and write the equation in standard form. Identify the coordinates (h, k) of the center (or in the case of a parabola, the vertex) of the conic section.

$$4x^2 - 32x + 99 = 9y^2 - 6y$$

$4x^2 - 32x - 9y^2 + 6y = -99$

$4(x^2 - 8x + 16) - 9(y^2 - \frac{2}{3}y + \frac{1}{9}) = -99 + 64 - 1$

$4(x-4)^2 - 9(y - \frac{1}{3})^2 = -36$

$\frac{(y - 1/3)^2}{4} - \frac{(x-4)^2}{9} = 1$

Hyperbola with center $(4, \frac{1}{3})$.

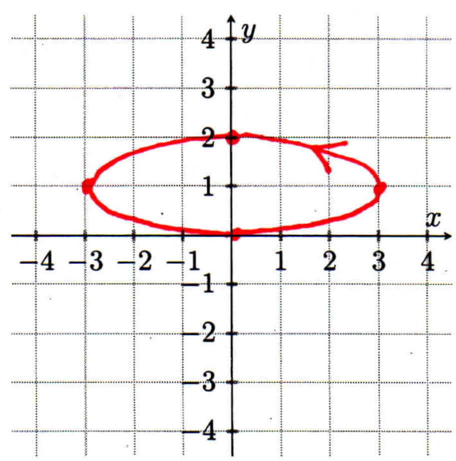
2. (1 mark) Convert the parametric equations $x = 3\cos\theta$ and $y = 1 + \sin\theta$ into an algebraic rectangular equation by eliminating the parameter and sketch its graph, including its orientation.

$\frac{x}{3} = \cos\theta$ and $y - 1 = \sin\theta$

Since $\cos^2\theta + \sin^2\theta = 1$, then

$\frac{x^2}{9} + (y-1)^2 = 1$

Ellipse with center $(0, 1)$.



3. Consider the set of parametric equations $x = \frac{1}{2}t^2 - t$ and $y = \frac{1}{2}t^2 + t$.

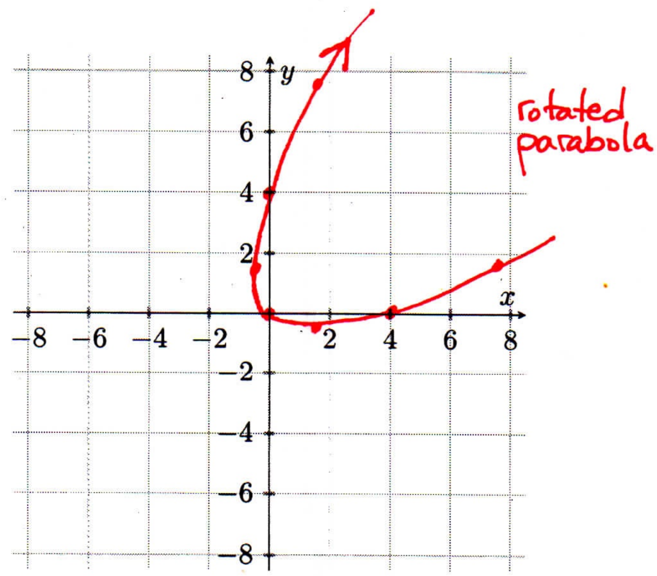
(a) (1 mark) Find the (x, y)-coordinates of any points of horizontal or vertical tangency to the curve.

$\frac{dx}{dt} = t - 1 = 0 \Rightarrow t = 1$ At $t=1$, $(x,y) = (-\frac{1}{2}, \frac{3}{2})$ V.T.

$\frac{dy}{dt} = t + 1 = 0 \Rightarrow t = -1$ At $t=-1$, $(x,y) = (\frac{3}{2}, -\frac{1}{2})$ H.T.

(b) (1 mark) Sketch the graph of the curve, including its orientation.

t	x	y	
-3	$15/2$	$3/2$	
-2	4	0	
-1	$3/2$	$-1/2$	H.T.
0	0	0	
1	$-1/2$	$3/2$	V.T.
2	0	4	
3	$3/2$	$15/2$	



(c) (1 mark) Eliminate the parameter and convert the parametric equations into a rectangular equation, one that does not involve roots. *Hint: Start by adding or subtracting the parametric equations.*

$x + y = t^2$

$y - x = 2t \Rightarrow t = \frac{y-x}{2}$

$\therefore\ x + y = \left(\frac{y-x}{2}\right)^2 \Rightarrow 4(x+y) = (y-x)^2$

or $x^2 - 2xy + y^2 - 4x - 4y = 0$

4. (2 marks) Find the arc length of the curve defined parametrically by $x = 4e^{t/2}$ and $y = e^t - t$ for $0 \le t \le 1$.

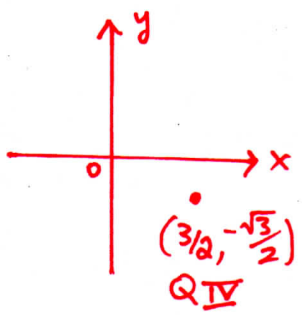
$$\frac{dx}{dt} = 2e^{t/2} \quad \text{and} \quad \frac{dy}{dt} = e^t - 1$$

$$\therefore ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(2e^{t/2})^2 + (e^t - 1)^2}\,dt = \sqrt{4e^t + e^{2t} - 2e^t + 1}\,dt$$

$$= \sqrt{e^{2t} + 2e^t + 1}\,dt = \sqrt{(e^t + 1)^2}\,dt = (e^t + 1)\,dt$$

$$\therefore s = \int_0^1 (e^t + 1)\,dt = \left[e^t + t\right]_0^1 = (e + 1) - (1 + 0) = e$$

5. (1 mark) Find two sets of polar coordinates (r, θ), with $0 \le \theta < 2\pi$, corresponding to the point having rectangular coordinates $(3/2, -\sqrt{3}/2)$.



$$r^2 = x^2 + y^2 = \left(\frac{3}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4} + \frac{3}{4} = 3 \Rightarrow r = \pm\sqrt{3}$$

$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}/2}{3/2} = -\frac{\sqrt{3}}{3} \Rightarrow \theta = \frac{5\pi}{6} \text{ or } \frac{11\pi}{6}$$

Since point is in QIV, then

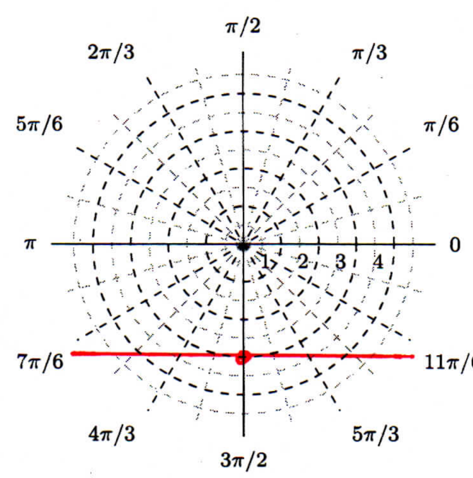
$$(r, \theta) = \left(\sqrt{3}, \frac{11\pi}{6}\right) \text{ or } \left(-\sqrt{3}, \frac{5\pi}{6}\right)$$

6. (1 mark) Convert the polar equation $r = -3\csc\theta$ to rectangular form and sketch its graph.

$$r = \frac{-3}{\sin\theta}$$

$$r\sin\theta = -3$$

$$\therefore y = -3$$



7. (3 marks) Consider the polar equation

$$r = 3 - 2\sin\theta.$$

(a) Convert the polar equation to a rectangular equation, one that does not involve roots.

$r^2 = 3r - 2r\sin\theta$

$x^2 + y^2 = 3r - 2y$

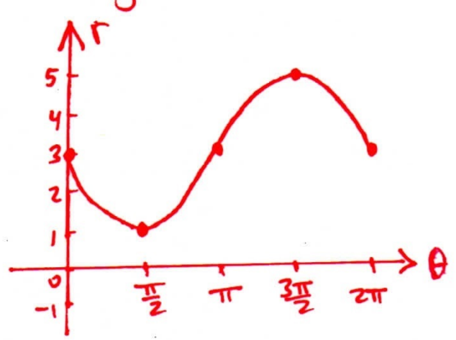
$3r = x^2 + y^2 + 2y$

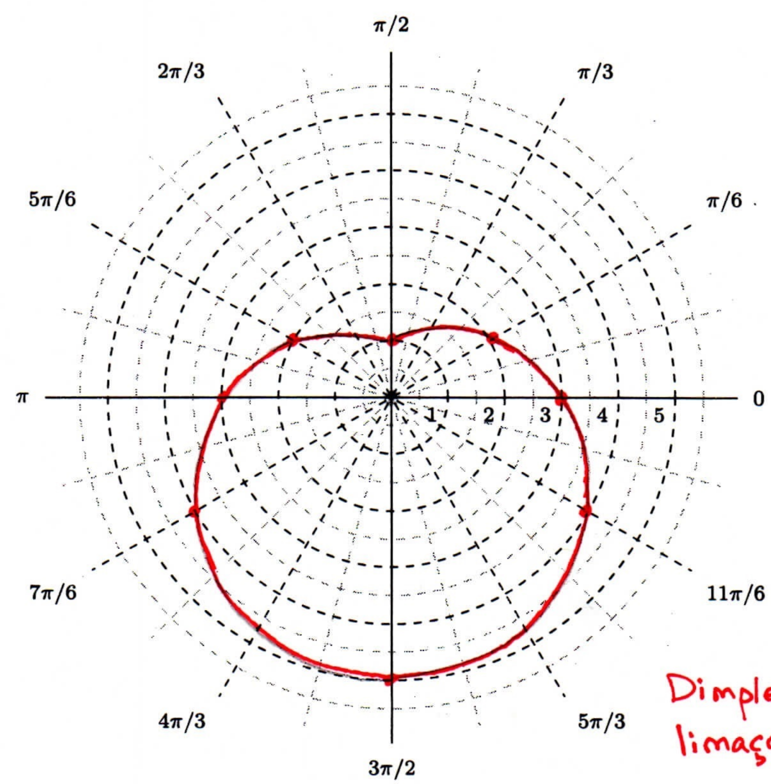
$9r^2 = (x^2 + y^2 + 2y)^2$

$9(x^2 + y^2) = (x^2 + y^2 + 2y)^2$

(b) Sketch the graph of the polar curve.

Rectangular Graph





Dimpled limaçon

Table of Values:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	3	2	1	2	3	4	5	4	3

8. (3 marks) Find the polar coordinates of any points of horizontal or vertical tangency, excluding the pole, to the polar curve $r = \cos^3\theta$ for $-\pi/2 \le \theta \le \pi/2$.

$x = r\cos\theta = \cos^3\theta\cos\theta = \cos^4\theta$

$\frac{dx}{d\theta} = -4\cos^3\theta\sin\theta = 0 \Rightarrow \cos\theta = 0$ or $\sin\theta = 0$

$\theta = -\frac{\pi}{2}, \frac{\pi}{2}$ $\quad$ $\theta = 0$

$(r = 0, \text{pole})$ $\quad$ $(r = 1)$

$y = r\sin\theta = \cos^3\theta\sin\theta$

$$\frac{dy}{d\theta} = \cos^3\theta(\cos\theta) + (-3\cos^2\theta\sin\theta)\sin\theta$$
$$= \cos^4\theta - 3\cos^2\theta\sin^2\theta$$
$$= \cos^2\theta(\cos^2\theta - 3\sin^2\theta)$$
$$= \cos^2\theta(1 - \sin^2\theta - 3\sin^2\theta)$$
$$= \cos^2\theta(1 - 4\sin^2\theta)$$
$$= \cos^2\theta(1 - 2\sin\theta)(1 + 2\sin\theta) = 0$$

$\Rightarrow \cos\theta = 0$, $\quad \sin\theta = \frac{1}{2}$, or $\sin\theta = -\frac{1}{2}$

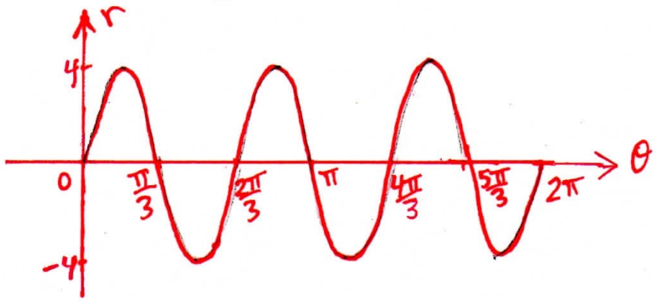
$\theta = -\frac{\pi}{2}, \frac{\pi}{2}$ $\quad \theta = \frac{\pi}{6}$ $\quad \theta = -\frac{\pi}{6}$

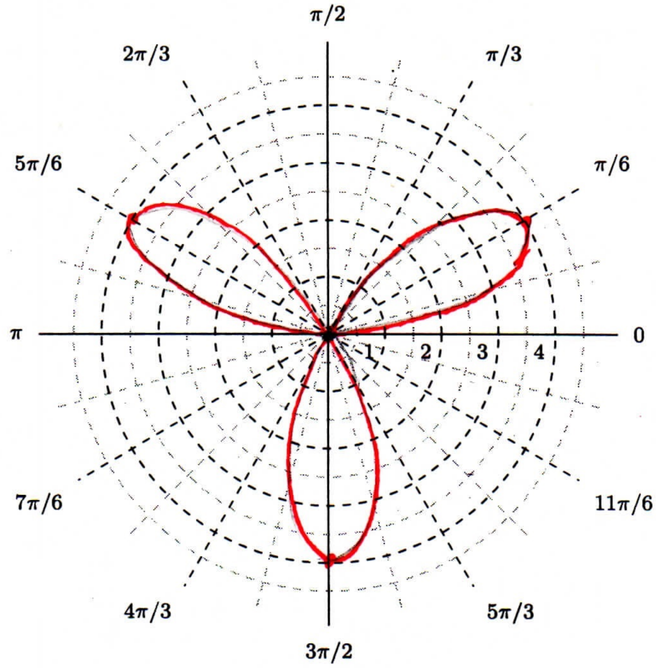
$(r = 0, \text{pole})$ $\quad (r = \frac{3\sqrt{3}}{8})$ $\quad (r = \frac{3\sqrt{3}}{8})$

$\therefore$ Vertical tangency at $(1, 0)$ and horizontal tangency at $(\frac{3\sqrt{3}}{8}, \frac{\pi}{6})$ and $(\frac{3\sqrt{3}}{8}, -\frac{\pi}{6})$.

9. (3 marks) Sketch the graph of the rose curve $r = 4\sin 3\theta$ and find the area of one petal.

Rectangular graph





One petal is traced for $0 \le \theta \le \frac{\pi}{3}$

Area $A = \frac{1}{2}\int_0^{\frac{\pi}{3}} (4\sin 3\theta)^2 \, d\theta$

$= 8\int_0^{\frac{\pi}{3}} \sin^2 3\theta \, d\theta$

$= 8\int_0^{\frac{\pi}{3}} \frac{1-\cos 6\theta}{2} \, d\theta$

$= 4\int_0^{\frac{\pi}{3}} (1-\cos 6\theta) \, d\theta$

$= 4\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{3}}$

$= 4\left(\frac{\pi}{3} - 0\right)$

$= \frac{4\pi}{3}$

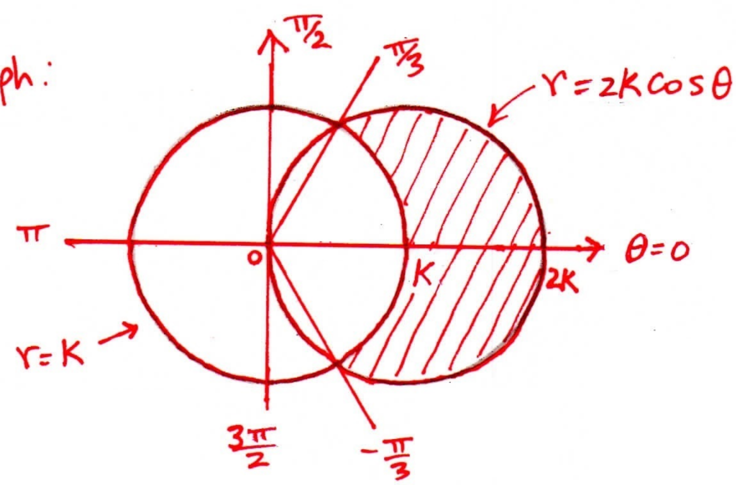
10. (3 marks) Find the area of the region inside the curve $r = 2k\cos\theta$ and outside the curve $r = k$, where $k > 0$ is a constant.

$r = k \Rightarrow r^2 = k^2 \Rightarrow x^2 + y^2 = k^2$ is circle of radius k centered at $(0,0)$

$r = 2k\cos\theta \Rightarrow r^2 = 2kr\cos\theta \Rightarrow x^2+y^2 = 2kx$

$\Rightarrow x^2 - 2kx + k^2 + y^2 = k^2 \Rightarrow (x-k)^2 + y^2 = k^2$ is circle of radius k centered at $(k, 0)$.

Graph:



Intersection:

$2k\cos\theta = k$

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$

$$\text{Area } A = 2\left[\frac{1}{2}\int_0^{\frac{\pi}{3}}(2k\cos\theta)^2\,d\theta - \frac{1}{2}\int_0^{\frac{\pi}{3}}k^2\,d\theta\right]$$

by symmetry

$$= \int_0^{\frac{\pi}{3}} 4k^2\cos^2\theta\,d\theta - \int_0^{\frac{\pi}{3}} k^2\,d\theta$$

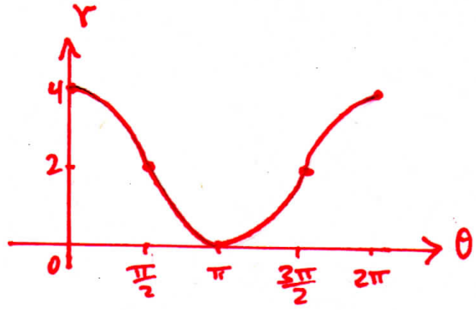
$$= \int_0^{\frac{\pi}{3}}\left[4k^2\left(\frac{1+\cos 2\theta}{2}\right) - k^2\right]d\theta$$

$$= k^2\int_0^{\frac{\pi}{3}}\left[1 + 2\cos 2\theta\right]d\theta$$

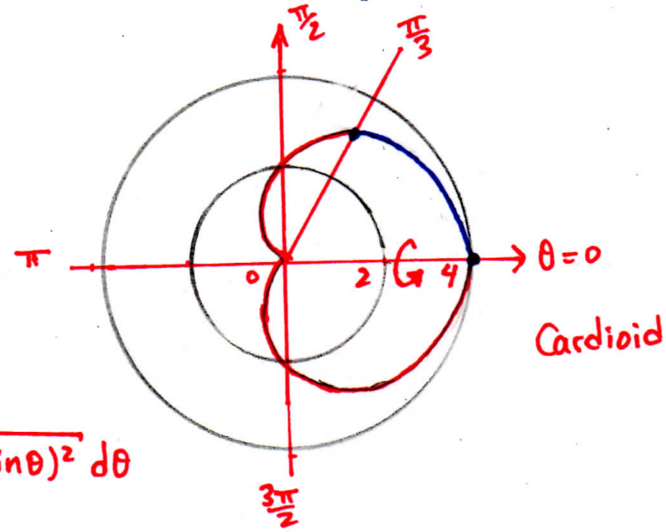
$$= k^2\left[\theta + \sin 2\theta\right]_0^{\pi/3}$$

$$= k^2\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$$

11. (3 marks) Find the area of the surface formed by revolving the polar curve $r = 2 + 2\cos\theta$ over the interval $[0, \pi/3]$ about the polar axis. Round your answer to three decimal places.



$\Rightarrow$



$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \sqrt{(2+2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$= \sqrt{8 + 8\cos\theta}\, d\theta = 2\sqrt{2}\sqrt{1+\cos\theta}\, d\theta$$

$$S = 2\pi\int_0^{\frac{\pi}{3}} r\sin\theta\, ds = 2\pi\int_0^{\frac{\pi}{3}} (2+2\cos\theta)\sin\theta \cdot 2\sqrt{2}\sqrt{1+\cos\theta}\, d\theta$$

$$= 8\pi\sqrt{2}\int_0^{\frac{\pi}{3}} (1+\cos\theta)\sin\theta\sqrt{1+\cos\theta}\, d\theta = -8\pi\sqrt{2}\int_0^{\frac{\pi}{3}} \underbrace{(1+\cos\theta)}_{u}{}^{3/2}\underbrace{(-\sin\theta)d\theta}_{du}$$

$$= -\frac{16\pi\sqrt{2}}{5}(1+\cos\theta)^{5/2}\Big|_0^{\pi/3} = -\frac{16\pi\sqrt{2}}{5}\left[\left(\frac{3}{2}\right)^{5/2} - 2^{5/2}\right]$$

$$= -\frac{16\pi\sqrt{2}}{5}\left(\frac{9\sqrt{3}}{4\sqrt{2}} - 4\sqrt{2}\right) = -\frac{16\pi\sqrt{2}}{5}\left(\frac{9\sqrt{3}-32}{4\sqrt{2}}\right)$$

$$= \frac{4\pi}{5}(32 - 9\sqrt{3}) \approx 41.247$$