**Mathematics 100 Test #3A** **Name:** SOLUTIONS

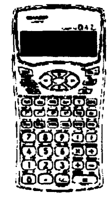
**Instructor: George Ballinger**
**Term: Fall, 2009**

**Mark:** $\overline{\phantom{00}}$ 25

The Sharp EL-531W calculator may be used on this test.

Show all of your work in the space provided.

The number of marks for each question is indicated in brackets.



1. Evaluate the following integrals.

(a) $\int \cos^2 3x \sin 3x \, dx$

$$= -\frac{1}{3}\int (\cos 3x)^2 (-3\sin 3x)\, dx$$

$$= -\frac{1}{3} \cdot \frac{1}{3}\cos^3 3x + C$$

[3]

$$= -\frac{1}{9}\cos^3 3x + C$$

(b) $\int x\sqrt{x+2}\, dx$

Let $u = x+2$
Then $du = dx$
and $x = u-2$

[3]

$$\int x\sqrt{x+2}\, dx = \int (u-2)\sqrt{u}\, du = \int (u-2)u^{1/2}\, du$$

$$= \int \left(u^{3/2} - 2u^{1/2}\right) du = \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$$

2. Consider the integral $\int_0^2 (x^3+4)\,dx$.

(a) Evaluate the integral by using the Fundamental Theorem of Calculus.

$$= \left[\frac{1}{4}x^4 + 4x\right]_0^2 = 4+8 = 12$$

[1]

(b) Evaluate the integral by using the limit of a Riemann sum definition of a definite integral:

$a=0,\ b=2$

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$$\int_0^2 (x^3+4)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n} = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^3+4\right)$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left(\frac{8i^3}{n^3}+4\right) = \lim_{n\to\infty} \frac{2}{n}\left[\frac{8}{n^3}\sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} 4\right]$$

[4]

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{8}{n^3}\cdot\frac{n^2(n+1)^2}{4} + 4n\right] = \lim_{n\to\infty} \frac{2}{n}\left[\frac{2(n+1)^2}{n} + 4n\right]$$

$$= \lim_{n\to\infty}\left[\frac{4(n+1)^2}{n^2} + 8\right] = \lim_{n\to\infty}\left[\frac{4(n^2+2n+1)}{n^2} + 8\right]$$

$$= \lim_{n\to\infty}\left[4\left(1+\frac{2}{n}+\frac{1}{n^2}\right) + 8\right] = 4+8 = 12$$

3. The acceleration of an object is given by $a(t) = \sin t$. If its initial velocity is 0, then find its velocity at time $t = \pi$.

$$V = \int a(t)\,dt = \int \sin t \, dt = -\cos t + C$$

$V = 0$ when $t = 0$

$\therefore \; 0 = -\cos 0 + C$

[3] $0 = -1 + C$

$C = 1$

$\therefore \; V = -\cos t + 1$

when $t = \pi$, $V = -\cos \pi + 1 = 1 + 1 = 2$

4. Suppose $f$ is a differentiable function that has an inverse $f^{-1}$. If $\underbrace{f(2) = 9}$, $f(9) = 2$, $f'(2) = -13$, and $f'(9) = -6$, then evaluate $(f^{-1})'(9)$.

$f^{-1}(9) = 2$

[1] $$(f^{-1})'(9) = \frac{1}{f'(f^{-1}(9))} = \frac{1}{f'(2)} = -\frac{1}{13}$$

5. Find the **exact** value (no decimals) of the definite integral $\displaystyle\int_e^{e^3} \frac{1}{x \ln x}\,dx$.

Let $u = \ln x$

then $du = \frac{1}{x}\,dx$

[3] $x = e \Rightarrow u = \ln e = 1$

$x = e^3 \Rightarrow u = \ln e^3 = 3$

$$\int_1^3 \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^3 = \ln 3 - \ln 1 = \ln 3$$

6. Use logarithmic differentiation to differentiate $y = x^{\sin x}$.

$$\ln y = \ln(x^{\sin x}) = (\sin x)\ln x$$

$$\frac{y'}{y} = (\sin x)\frac{1}{x} + (\cos x)\ln x$$

[3]

$$y' = x^{\sin x}\left(\frac{\sin x}{x} + (\cos x)\ln x\right)$$

7. Find all critical numbers of the following functions.

(a) $f(x) = \frac{x^6}{e^{3x}}$

$$f'(x) = \frac{e^{3x}(6x^5) - x^6(3e^{3x})}{(e^{3x})^2}$$

$$= \frac{3x^5e^{3x}(2-x)}{(e^{3x})^2} = \frac{3x^5(2-x)}{e^{3x}}$$

[3]

$\therefore$ $f'(x) = 0$ when $x = 0$ or $x = 2$

(b) $F(x) = \int_1^x 5^t (t+2)^2 \, dt$

$$F'(x) = 5^x(x+2)^2$$

[1]

$F'(x) = 0$ when $x = -2$