CAMOSUN COLLEGE

Mathematics 100 Test #3B

Name: SOLUTIONS

Instructor: George Ballinger
Term: Winter, 2010

Mark: $\overline{}$ 25

> The Sharp EL-531W calculator may be used on this test.
>
> Show all of your work in the space provided.
>
> The number of marks for each question is indicated in brackets.

1. Evaluate the following integrals.

(a) $\int \frac{x^3}{(x^4-4)^6}\,dx = \int x^3(x^4-4)^{-6}\,dx = \frac{1}{4}\int \underbrace{(x^4-4)}_{u}{}^{-6}\underbrace{(4x^3)\,dx}_{du}$

$= -\frac{1}{20}(x^4-4)^{-5} + C = \frac{-1}{20(x^4-4)^5} + C$

[3]

(b) $\int x\sqrt[3]{x-2}\,dx$ Let $u = x-2$ then $du = dx$ and $x = u+2$

$= \int (u+2)u^{\frac{1}{3}}\,du = \int \left(u^{4/3} + 2u^{1/3}\right)du = \frac{3}{7}u^{7/3} + \frac{3}{2}u^{4/3} + C$

[3] $= \frac{3}{7}(x-2)^{7/3} + \frac{3}{2}(x-2)^{4/3} + C$

2. Find the **exact** value of the definite integral $\displaystyle\int_{\pi/3}^{\pi/2} \sin^3\frac{x}{2}\cos\frac{x}{2}\,dx$.

Let $u = \sin\frac{x}{2}$

then $du = \frac{1}{2}\cos\frac{x}{2}\,dx$

$$= 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \underbrace{\left(\sin\frac{x}{2}\right)}_{u}^3 \cdot \underbrace{\frac{1}{2}\cos\frac{x}{2}\,dx}_{du} = \frac{1}{2}\left[\sin^4\frac{x}{2}\right]_{\pi/3}^{\pi/2} = \frac{1}{2}\left[\sin^4\frac{\pi}{4} - \sin^4\frac{\pi}{6}\right]$$

[3]

$$= \frac{1}{2}\left[\left(\frac{1}{\sqrt{2}}\right)^4 - \left(\frac{1}{2}\right)^4\right] = \frac{1}{2}\left[\frac{1}{4} - \frac{1}{16}\right] = \frac{1}{2}\left(\frac{3}{16}\right) = \frac{3}{32}$$

3. Let $f(x) = e^{-x/2}$.
   (a) Find the average value of $f(x)$ on the interval $[0, 3]$. Round your answer to four decimal places.
   (b) Find the value(s) of x in the interval $[0, 3]$ for which the function equals its average value. Round your answer(s) to four decimal places.
   (c) Sketch the graph of $f(x)$ on the interval $[0, 3]$ and label the average value and corresponding x-value(s) on your graph.

a) Avg value $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \dfrac{1}{3-0}\int_0^3 e^{-x/2}\,dx = \frac{1}{3}\int_0^3 e^{-x/2}\,dx$

$$= -\frac{2}{3}\left[e^{-x/2}\right]_0^3 = -\frac{2}{3}\left[e^{-3/2} - 1\right] \approx 0.5179$$

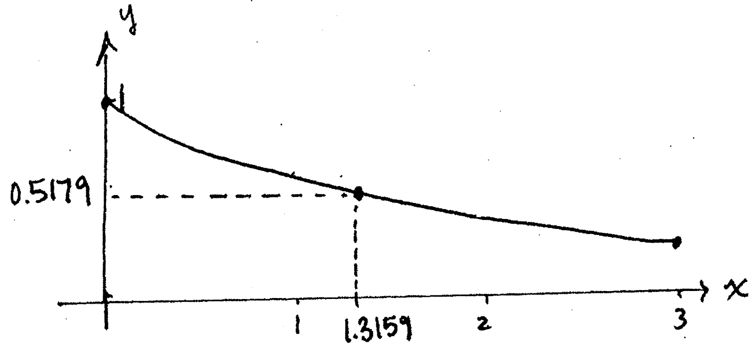
[5]

b) $e^{-x/2} = 0.5179$

$-\frac{x}{2} = \ln 0.5179$

$x = -2\ln 0.5179 \approx 1.3159$

c)



4. If the Trapezoidal Rule were used to approximate the integral $\int_2^4 \frac{1}{\sqrt{x}}\,dx$ with $n = 24$, then calculate the **exact** value of $f(x_3)$ in the formula for the Trapezoidal Rule. Do not compute the entire approximation.

$$\boxed{\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{n-1})+f(x_n)]}$$

[2]

$$\Delta x = \frac{b-a}{n} = \frac{4-2}{24} = \frac{2}{24} = \frac{1}{12}$$

$$x_i = a + i\Delta x = 2 + \frac{i}{12}$$

$$x_3 = 2 + \frac{3}{12} = \frac{9}{4} \qquad \therefore\ f(x_3) = f\left(\frac{9}{4}\right) = \frac{1}{\sqrt{9/4}} = \frac{2}{3}$$

5. Evaluate $\int_{-5}^{5} (x+5)^3\,dx$ by using the limit of a Riemann sum definition of a definite integral:

$$\Delta x = \frac{5-(-5)}{n} = \frac{10}{n}$$

[3]

$$\int_{-5}^{5} (x+5)^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-5+\frac{10i}{n}\right)\frac{10}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(-5+\frac{10i}{n}\right)+5\right]^3 \frac{10}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{10i}{n}\right)^3 \cdot \frac{10}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{10000\, i^3}{n^4}$$

$$= \lim_{n\to\infty} \frac{10000}{n^4} \sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} \frac{10000}{n^4} \cdot \frac{n^2(n+1)^2}{4}$$

$$= \lim_{n\to\infty} \frac{2500}{n^2} \cdot (n^2+2n+1)$$

$$= \lim_{n\to\infty} 2500\left(1+\cancelto{0}{\frac{2}{n}}+\cancelto{0}{\frac{1}{n^2}}\right)$$

$$= 2500$$

6. A particle moves along the x-axis at a velocity $v(t) = 6\sec 2t \tan 2t$ for $t \geq 0$. At time $t = 0$, its position is $x = 4$. Find the position of the particle at time $t = 1/2$. Round your answer to three decimal places.

[3]

$$x(t) = \int v(t)\,dt = \int 6\sec 2t \tan 2t \, dt = 3\sec 2t + C$$

$$x(0) = 4 \Rightarrow 3\sec(2\cdot 0) + C = 4$$

$$3 + C = 4$$

$$C = 1$$

$$\therefore \; x(t) = 3\sec 2t + 1$$

$$\therefore \; x(\tfrac{1}{2}) = 3\sec 2(\tfrac{1}{2}) + 1 = 3\sec 1 + 1 \approx 6.552$$

7. Use logarithmic differentiation to differentiate $y = x^{\cos x}$.

[3]

$$\ln y = \ln x^{\cos x} = \cos x \ln x$$

$$\frac{y'}{y} = (\cos x)\frac{1}{x} + (-\sin x)\ln x$$

$$y' = y\left[\frac{\cos x}{x} - \sin x \ln x\right]$$

$$y' = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \ln x\right]$$