**Mathematics 100 Test #3B** **Name:** SOLUTIONS

**Instructor: George Ballinger** **Section:** ____________

**Term: Fall, 2017**

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| The Sharp EL-531 calculator may be used on this test. |
| Show all of your work in the space provided. |
| The number of marks for each question is indicated in brackets. |

**Mark:** $\frac{\quad}{25}$

1. Find the following integrals.

(a) $\int \csc^2 6x \cot 6x \, dx$

[2]

$$= -\frac{1}{6}\int \underbrace{\cot 6x}_{u} \underbrace{(-6\csc^2 6x)\,dx}_{du} = -\frac{1}{12}\cot^2 6x + C$$

(b) $\int_4^6 \frac{x^2+2}{x-3}\,dx$

[3]

$$\begin{array}{r} x+3 \\ x-3\,\overline{)\,x^2 \quad +2} \\ \underline{x^2-3x} \\ 3x+2 \\ \underline{3x-9} \\ 11 \end{array}$$

$$\int_4^6 \left(x+3+\frac{11}{x-3}\right)dx = \left[\tfrac{1}{2}x^2+3x+11\ln|x-3|\right]_4^6$$

$$=(36+11\ln 3)-(20+\cancel{11\ln 1}^{\,0})$$

$$=16+11\ln 3$$

OR Let $u=x-3 \rightarrow x=u+3$

$du=dx$

$x=4 \Rightarrow u=1$

$x=6 \Rightarrow u=3$

$$\int_1^3 \frac{(u+3)^2+2}{u}\,du = \int_1^3 \frac{u^2+6u+11}{u}\,du$$

$$=\int_1^3\left(u+6+\frac{11}{u}\right)du = \left[\tfrac{1}{2}u^2+6u+11\ln|u|\right]_1^3$$

$$=\left(\tfrac{45}{2}+11\ln 3\right)-\left(\tfrac{13}{2}+\cancel{11\ln 1}^{\,0}\right) = 16+11\ln 3$$

(c) $\int x^7\sqrt{x^4+1}\,dx$

[4]

Let $u=x^4+1 \Rightarrow x^4=u-1$

$du=4x^3dx \Rightarrow x^3dx=\frac{1}{4}du$

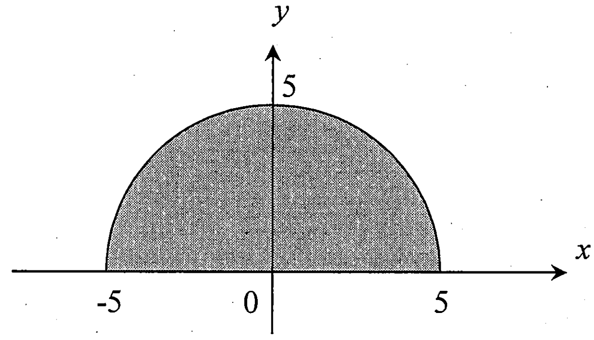
(OR let $u=\sqrt{x^4+1}$ ...)

$$\int x^4(x^4+1)^{1/2}x^3dx = \int (u-1)u^{1/2}\cdot\tfrac{1}{4}du = \tfrac{1}{4}\int\left(u^{3/2}-u^{1/2}\right)du$$

$$=\tfrac{1}{4}\left[\tfrac{2}{5}u^{5/2}-\tfrac{2}{3}u^{3/2}\right]+C = \tfrac{1}{10}(x^4+1)^{5/2}-\tfrac{1}{6}(x^4+1)^{3/2}+C$$

OR $\frac{1}{30}(x^4+1)^{3/2}(3x^4-2)+C$, if factored

2. Set up a definite integral representing the area of the semi-circular region shown. What is the value of the integral?




[2]

$$A = \int_{-5}^{5} \sqrt{25-x^2}\,dx = \frac{25\pi}{2}$$

3. Find the average value of $f(x) = \dfrac{e^{1/x}}{x^2}$ over the interval $[1/2, 1]$.

$$\text{Avg. value} = \frac{1}{1-\frac{1}{2}} \int_{\frac{1}{2}}^{1} \frac{e^{\frac{1}{x}}}{x^2}\,dx = -2\int_{\frac{1}{2}}^{1} \underbrace{e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)dx}_{\text{form } e^u du}$$

[3]

$$= -2e^{\frac{1}{x}}\Big|_{\frac{1}{2}}^{1} = -2\left(e - e^2\right) = 2e^2 - 2e$$

4. Evaluate $\displaystyle\int_{-2}^{6} (x+2)^3\,dx$ by using the limit of a Riemann sum definition of a definite integral.

$$\boxed{\begin{aligned} f(x) &= (x+2)^3 \\ \Delta x &= \frac{b-a}{n} = \frac{6-(-2)}{n} = \frac{8}{n}\end{aligned}}$$

$$\int_{-2}^{6} (x+2)^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-2+\frac{8i}{n}\right)\frac{8}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(-2+\frac{8i}{n}\right)+2\right]^3 \frac{8}{n}$$

[3]

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8i}{n}\right)^3 \frac{8}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4096\, i^3}{n^4}$$

$$= \lim_{n\to\infty} \frac{4096}{n^4} \sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} \frac{4096}{n^4} \cdot \frac{n^2(n+1)^2}{4}$$

$$= \lim_{n\to\infty} \frac{1024\left(n^2+2n+1\right)}{n^2}$$

$$= \lim_{n\to\infty} \left(1024 + \cancelto{0}{\frac{2048}{n}} + \cancelto{0}{\frac{1024}{n^2}}\right)$$

$$= 1024$$

5. Use logarithmic differentiation to find the derivative of $y = (\ln x)^x$.

$$\ln y = \ln (\ln x)^x = x \ln (\ln x)$$

$$\frac{y'}{y} = x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} + 1 \cdot \ln (\ln x)$$

$$y' = y\left(\frac{1}{\ln x} + \ln (\ln x)\right)$$

[3]

$$y' = (\ln x)^x \left(\frac{1}{\ln x} + \ln (\ln x)\right)$$

6. Find the particular solution, $f(x)$, of the differential equation with the given initial condition.

$f'(x) = 3^x$, $f(0) = 1$

$$f(x) = \int 3^x dx = \int e^{(\ln 3)x} dx = \frac{1}{(\ln 3)} e^{(\ln 3)x} + C$$

$$= \frac{3^x}{\ln 3} + C$$

[3]

$$f(0) = 1 \Rightarrow \frac{1}{\ln 3} + C = 1 \Rightarrow C = 1 - \frac{1}{\ln 3}$$

$$\therefore f(x) = \frac{3^x}{\ln 3} + 1 - \frac{1}{\ln 3} \quad \text{or} \quad \frac{3^x - 1}{\ln 3} + 1$$

7. Define $F(x) = \int_{-\pi/6}^{x} \sin^3 \theta \, d\theta$. Evaluate $F'(\pi/6)$ by using the Second Fundamental Theorem of Calculus.

$$F'(x) = \sin^3 x$$

$$F'(\pi/6) = \sin^3 \frac{\pi}{6} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

[2]