CAMOSUN

Name: SOLUTIONS

Mark: $\overline{}$ 25

MATH 100 (Fall, 2022)

Test 3B

1. (4 marks) Evaluate $\displaystyle\int_1^5 6(x-1)^2\,dx$ by using the limit definition of a definite integral.

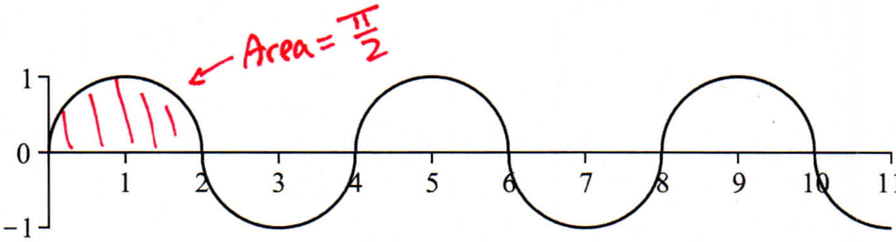
$a=1,\ b=5,\ \Delta x = \dfrac{b-a}{n} = \dfrac{5-1}{n} = \dfrac{4}{n},\quad f(x) = 6(x-1)^2$

$$\int_1^5 6(x-1)^2dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{4i}{n}\right)\frac{4}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[6\left(1+\frac{4i}{n}-1\right)^2\right]\frac{4}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} 6\left(\frac{4i}{n}\right)^2\left(\frac{4}{n}\right) = \lim_{n\to\infty}\frac{384}{n^3}\sum_{i=1}^{n} i^2 = \lim_{n\to\infty}\frac{384}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty}\frac{64(2n^2+3n+1)}{n^2} = \lim_{n\to\infty} 64\left(2+\frac{3}{n}+\frac{1}{n^2}\right) = 128$$

2. (2 marks) Consider the periodic wave function $y = f(x)$ consisting of unit semicircles, as shown in the graph below. Evaluate the following definite integrals.



$\displaystyle\int_0^2 f(x)\,dx = \underline{\frac{\pi}{2}}$; $\displaystyle\int_2^0 f(x)\,dx = \underline{-\frac{\pi}{2}}$; $\displaystyle\int_0^4 f(x)\,dx = \underline{0}$; $\displaystyle\int_0^4 |f(x)|\,dx = \underline{\pi}$

3. Let $f(x) = \dfrac{1}{x}$.

(a) (2 marks) Use Simpson's rule with $n = 4$ to approximate $\int_1^5 f(x)\,dx$.

$$\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = 1$$

$$\int_1^5 f(x)\,dx \approx \frac{5-1}{3(4)}\left[f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)\right]$$

$$= \frac{1}{3}\left[1 + 2 + \frac{2}{3} + 1 + \frac{1}{5}\right] = \frac{73}{45}$$

(b) (1 mark) Find the exact value of $\int_1^5 f(x)\,dx$.

$$\int_1^5 \frac{1}{x}\,dx = \ln|x|\Big|_1^5 = \ln 5 - \ln 1 = \ln 5$$

(c) (1 mark) Find the average value of $f(x)$ on the interval $[1, 5]$.

$$\text{Avg. Value} = \frac{1}{5-1}\int_1^5 \frac{1}{x}\,dx = \frac{1}{4}\ln 5$$

(d) (1 mark) Find the exact value of c guaranteed to exist according to the Mean Value Theorem for Integrals for f on the interval $[1, 5]$.

$$f(c) = \frac{\ln 5}{4} \Rightarrow \frac{1}{c} = \frac{\ln 5}{4} \Rightarrow c = \frac{4}{\ln 5}$$

4. (4 marks) Integrate $\int 9x\sqrt{3x+1}\,dx$.

Let $u = 3x+1$. Then

$x = \frac{u-1}{3}$ and $dx = \frac{1}{3}du$

$$\int 9\left(\frac{u-1}{3}\right)u^{1/2}\cdot\frac{1}{3}du$$

$$= \int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(3x+1)^{5/2} - \frac{2}{3}(3x+1)^{3/2} + C$$

OR

Let $u = \sqrt{3x+1}$. Then

$x = \frac{u^2-1}{3}$ and $dx = \frac{2}{3}u\,du$

$$\int 9\left(\frac{u^2-1}{3}\right)u\cdot\frac{2}{3}u\,du$$

$$= \int \left(2u^4 - 2u^2\right)du$$

$$= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C$$

$$= \frac{2}{5}(3x+1)^{5/2} - \frac{2}{3}(3x+1)^{3/2} + C$$

OR $\frac{2}{15}(3x+1)^{3/2}(9x-2)+C$ if factored

5. (2 marks) Differentiate $f(x) = 4xe^{-5x^2}$.

$$f'(x) = 4x\cdot e^{-5x^2}(-10x) + 4\cdot e^{-5x^2} = -40x^2e^{-5x^2} + 4e^{-5x^2}$$

OR $-4e^{-5x^2}(10x^2-1)$ if factored

6. (2 marks) Find and simplify $f'(4)$ if $f(x) = \log_2(2^x+2)$.

$$f'(x) = \frac{1}{(\ln 2)(2^x+2)}\cdot(\ln 2)2^x = \frac{2^x}{2^x+2}$$

$$\therefore f'(4) = \frac{16}{18} = \frac{8}{9}$$

7. (3 marks) Find the particular solution, $f(x)$, of the differential equation with the given initial condition.

$$f'(x) = \sec x, \;\; f(\pi) = 3$$

$$f(x) = \int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$f(\pi) = 3 \Rightarrow 3 = \ln|\sec\pi + \tan\pi| + C = \ln|-1| + C = \ln 1 + C = 0 + C = C$$

$$\therefore \; C = 3$$

$$\therefore \; f(x) = \ln|\sec x + \tan x| + 3$$

8. (3 marks) Find the slope of the tangent line to the curve defined implicitly by the equation $y^y = x^x$ at the point $(1/4, 1/2)$. Round your answer to three decimal places.

$$\ln y^y = \ln x^x \Rightarrow y \ln y = x \ln x$$

$$\therefore \; y \cdot \frac{y'}{y} + y' \cdot \ln y = x \cdot \frac{1}{x} + 1 \cdot \ln x$$

$$y'(1 + \ln y) = 1 + \ln x$$

$$y' = \frac{1 + \ln x}{1 + \ln y}$$

$$\therefore \; \text{slope} = \frac{1 + \ln\frac{1}{4}}{1 + \ln\frac{1}{2}} \approx -1.259$$