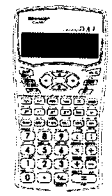
Mathematics 100 Test #2B **Name:** SOLUTIONS

Instructor: George Ballinger
Term: Winter, 2010 **Mark:** $\frac{\quad}{25}$

The Sharp EL-531W calculator may be used on this test.



You may not use L'Hôpital's Rule when evaluating limits.

Show all of your work in the space provided.

The number of marks for each question is indicated in brackets.

1. Find each limit and show your work. If the limit does not exist, then answer ∞ or $-\infty$ if applicable.

(a) $\lim\limits_{x\to-\infty}\dfrac{x^2+3x}{x^2} = \lim\limits_{x\to-\infty}\left(1+\frac{3}{x}\right) = 1$

[1]

(b) $\lim\limits_{x\to-\infty}\dfrac{x^2+3x}{x} = \lim\limits_{x\to-\infty}(x+3) = -\infty$

[1]

(c) $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+3x}}{x} = \lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+3x}}{-\sqrt{x^2}} = \lim\limits_{x\to-\infty}-\sqrt{\dfrac{x^2+3x}{x^2}} = \lim\limits_{x\to-\infty}-\sqrt{1+\frac{3}{x}} = -1$

[1]

2. Find the slope at the point $(-1, 2)$ of the tangent line to the curve defined implicitly by

$$-3x^4+3xy+6x=2y^2-23.$$

$$\frac{d}{dx}\left[-3x^4+3xy+6x\right] = \frac{d}{dx}\left[2y^2-23\right]$$

$$-12x^3+3x\frac{dy}{dx}+3y+6 = 4y\frac{dy}{dx}$$

[3]

$$(3x-4y)\frac{dy}{dx} = 12x^3-3y-6$$

$$\frac{dy}{dx} = \frac{12x^3-3y-6}{3x-4y}$$

At $(-1,2)$ slope $= \dfrac{dy}{dx} = \dfrac{12(-1)^3-3(2)-6}{3(-1)-4(2)} = \dfrac{-24}{-11} = \dfrac{24}{11}$

3. Let $g(\theta) = \theta \sin\theta + \cos\theta + 1$.

(a) Determine whether Rolle's Theorem can be applied to g on the closed interval $[0, 2\pi]$. If Rolle's Theorem can be applied, then find all values of c in the open interval $(0, 2\pi)$ that are predicted to exist according to Rolle's Theorem. If Rolle's Theorem cannot be applied, then explain why not.

(b) Find the absolute minimum and maximum values of g on the interval $[0, 2\pi]$. Round your answers to three decimal places.

a) g is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$ and $g(0) = 2 = g(2\pi)$ $\therefore$ Rolle's Theorem can be applied.

$$g'(\theta) = \theta\cos\theta + \sin\theta - \sin\theta = \theta\cos\theta$$

$$g'(c) = 0 \implies c\cos c = 0 \implies c = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ in } (0, 2\pi).$$

[5] b) Critical #'s are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Endpoints are 0 and 2π

$g(0) = 2$

$g(\frac{\pi}{2}) = \frac{\pi}{2} + 1 \approx 2.571$ ← Max value of g

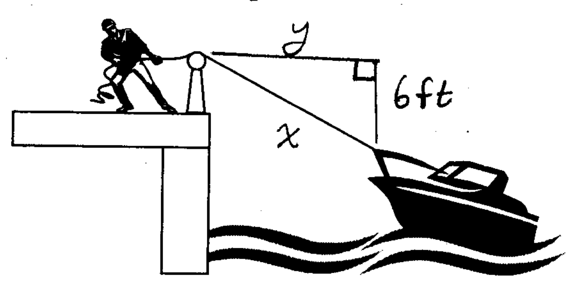
$g(\frac{3\pi}{2}) = -\frac{3\pi}{2} + 1 \approx -3.712$ ← Min value of g

$g(2\pi) = 2$

4. A boat is pulled into a dock via a rope with one end attached to the bow of a boat and the other end held by a man 6 ft above the bow of the boat. If the man pulls the rope at a constant rate of 2 ft/sec, how fast is the boat moving toward the dock when 12 ft of rope is out? Round your answer to three decimal places

Given: $\frac{dx}{dt} = -2$ ft/sec

Find: $\frac{dy}{dt}$ when $x = 12$ ft



$$y^2 + 6^2 = x^2$$

$$\frac{d}{dt}[y^2 + 36] = \frac{d}{dt}[x^2]$$

[4] $$2y\frac{dy}{dt} = 2x\frac{dx}{dt} \implies \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} = -2\left(\frac{x}{y}\right)$$

when $x = 12$, $y = \sqrt{x^2 - 6^2} = \sqrt{12^2 - 6^2} = 6\sqrt{3}$

$$\therefore \frac{dy}{dt} = -2\left(\frac{12}{6\sqrt{3}}\right) = \frac{-4\sqrt{3}}{3} \approx -2.309 \text{ ft/sec.}$$

$\therefore$ Speed is 2.309 ft/sec toward dock.

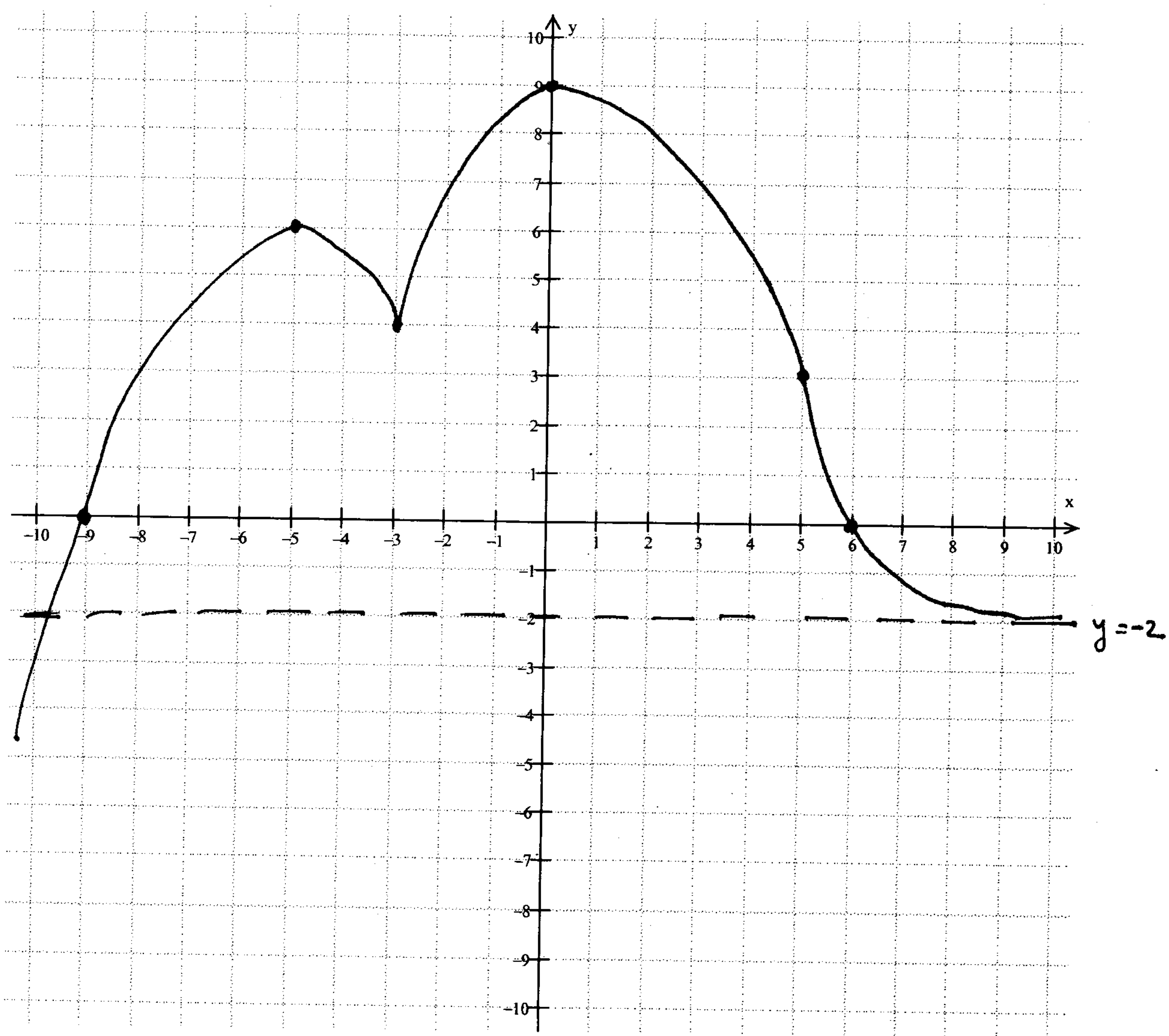
5. Sketch the graph of a **continuous** function f that clearly satisfies all of the following characteristics. Your graph should clearly show the increasing, decreasing and concave structure of f as well as other features such as relative extrema and asymptotes.

f has an x-intercept at $x=-9$ f has an x-intercept at $x=6$
f has a y-intercept at $y=9$
$f(-5)=6$ $f(-3)=4$ $f(5)=3$
$\lim_{x\to\infty} f(x)=-2$ $\lim_{x\to-\infty} f(x)=-\infty$
$f'(-5)=0$ $f'(-3)$ is undefined $f'(0)=0$
$f''(5)=0$

Interval	$(-\infty,-5)$	$(-5,-3)$	$(-3,0)$	$(0,\infty)$
Sign of $f'(x)$	+	−	+	−

Interval	$(-\infty,-3)$	$(-3,5)$	$(5,\infty)$
Sign of $f''(x)$	−	−	+

[4]



6. Let $f(x) = x^3 + 2x - 1$. If Newton's Method were used with an initial estimate $x_1 = 0.5$ to approximate a zero of f, then compute the second estimate x_2. Round your answer to four decimal places.

$$f'(x) = 3x^2 + 2$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \frac{0.125}{2.75} \approx 0.4545$$

[2]

7. The period (measured in seconds) of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum in feet and g is the constant of acceleration due to gravity.

(a) Solve for L in this equation.

(b) Express the differential dL in terms of T, dT and the constant g.

(c) If $g = 32$ ft/sec^2, then use differentials to estimate the maximum absolute error in calculating the length of a pendulum whose period is measured to be 2.6 sec with a possible error of 0.02 sec. Round your answer to four decimal places.

a) $\sqrt{\frac{L}{g}} = \frac{T}{2\pi} \Rightarrow \frac{L}{g} = \frac{T^2}{4\pi^2} \Rightarrow L = \left(\frac{g}{4\pi^2}\right)T^2$

b) $dL = 2\left(\frac{g}{4\pi^2}\right)T\,dT = \left(\frac{g}{2\pi^2}\right)T\,dT$

[4]

c) $T = 2.6$ sec, $dT = \pm 0.02$ sec

$\therefore \Delta L \approx dL = \left(\frac{32}{2\pi^2}\right)(2.6)(\pm 0.02) \approx \pm 0.0843$ ft