Mathematics 100 Test #1A **Name:** SOLUTIONS

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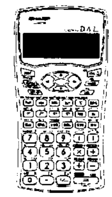
Mark: $\frac{\quad}{25}$

The Sharp EL-531W calculator may be used on this test.

You may not use L'Hôpital's Rule when evaluating limits.

Show all of your work in the space provided.

The number of marks for each question is indicated in brackets.



1. Give an example of a function $f(x)$ that is continuous everywhere but is not differentiable at $x = 0$. [Note: There are many correct answers.]

[1] $f(x) = |x|$ or $\sqrt[3]{x}$ (for example)

2. If $f(x) = \begin{cases} 2x-8, & x>3 \\ 3 & x=3 \\ 2x+8, & x<3 \end{cases}$, then determine $\lim\limits_{x\to 3^-} f(x)$.

[1] $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (2x+8) = 2(3)+8 = 6+8 = 14$

3. Find the x-values at which $f(x) = \dfrac{x^2-7x+12}{x^2-9}$ is not continuous and determine which, if any, of these discontinuities is removable.

$$f(x) = \frac{(x-4)(x-3)}{(x+3)(x-3)} = \frac{x-4}{x+3}$$

f is not continuous (not defined) at $x = \pm 3$.

[2] f has a removable discontinuity at $x = 3$.

4. Evaluate the limits. If they do not exist, then determine whether they are ∞ or $-\infty$ or neither.

(a) $\lim\limits_{x \to 0} \dfrac{6x^3 - 10x}{x}$

[1] $= \lim\limits_{x \to 0} \left(6x^2 - 10\right) = -10$

(b) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sin x}{x}$

[1] $= \dfrac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} = \dfrac{1}{\frac{\pi}{2}} = \dfrac{2}{\pi}$

(c) $\lim\limits_{x \to 2^+} \dfrac{3}{4 - 2x}$

[1] $= \dfrac{3}{0^-} = -\infty$

(d) $\lim\limits_{x \to 0} \dfrac{\tan x}{4x \cos x}$

$= \lim\limits_{x \to 0} \dfrac{\sin x}{4x \cos^2 x}$

[2] $= \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x \to 0} \dfrac{1}{4\cos^2 x}$

$= 1 \cdot \dfrac{1}{4}$

$= \dfrac{1}{4}$

5.

(a) State the limit definition of the derivative of $f(x)$.

[1] $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

(b) Use the *limit definition* of the derivative to find $f'(x)$, where $f(x) = \sqrt{2x+1}$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{2(x+\Delta x)+1} - \sqrt{2x+1}}{\Delta x}$$

[3]

$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{2x+2\Delta x+1} - \sqrt{2x+1}\right)}{\Delta x} \cdot \frac{\left(\sqrt{2x+2\Delta x+1} + \sqrt{2x+1}\right)}{\left(\sqrt{2x+2\Delta x+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+2\Delta x+1) - (2x+1)}{\Delta x\left(\sqrt{2x+2\Delta x+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x\left(\sqrt{2x+2\Delta x+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{2}{\sqrt{2x+2\Delta x+1} + \sqrt{2x+1}}$$

$$= \frac{2}{\sqrt{2x+1} + \sqrt{2x+1}}$$

$$= \frac{2}{2\sqrt{2x+1}}$$

$$= \frac{1}{\sqrt{2x+1}}$$

6. Consider the curve $y = x^3 - 2x^2 + 5x - 6$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ by using differentiation rules.

[2]

$$\frac{dy}{dx} = 3x^2 - 4x + 5$$

$$\frac{d^2y}{dx^2} = 6x - 4$$

(b) Find the equation of the tangent line to the curve at $x = 1$. Express your answer in the slope-intercept form $y = mx + b$.

At $x = 1$, $y = -2$ and $\frac{dy}{dx} = 4 \leftarrow$ slope m

[2]

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = 4(x - 1)$$

$$y + 2 = 4x - 4$$

$$y = 4x - 6$$

7. The position of an object in meters after t seconds is given by $s(t) = 4\sqrt[3]{t+7}$. Calculate the instantaneous velocity (in m/s) at $t = 20$ seconds.

[3]

$$s(t) = 4(t+7)^{1/3}$$

$$v(t) = s'(t) = \frac{4}{3}(t+7)^{-2/3}$$

$$v(20) = \frac{4}{3}(20+7)^{-2/3} = \frac{4}{3}(27)^{-2/3}$$

$$= \frac{4}{3} \cdot \frac{1}{(\sqrt[3]{27})^2} = \frac{4}{3} \cdot \frac{1}{9} = \frac{4}{27} \text{ m/s}$$

8. Use the derivative rules to calculate the derivatives of the following functions. Simplify your answers.

(a) $y = 3x^2 \sin x$

$$y' = 3x^2 \cos x + 6x \sin x$$

[1]

(b) $f(x) = 2\tan^4(5x^3)$

$$f'(x) = 8\tan^3(5x^3) \cdot \sec^2(5x^3) \cdot 15x^2$$

[2]

$$= 120x^2 \tan^3(5x^3)\sec^2(5x^3)$$

(c) $g(t) = \dfrac{2t^2 - 1}{t^2 + 3}$

$$g'(t) = \frac{(t^2+3)(4t) - (2t^2-1)(2t)}{(t^2+3)^2}$$

[2]

$$= \frac{4t^3 + 12t - 4t^3 + 2t}{(t^2+3)^2}$$

$$= \frac{14t}{(t^2+3)^2}$$