Mathematics 100 Test #1B **Name:** SOLUTIONS

Instructor: George Ballinger
Term: Winter, 2010 **Mark:** ___ / 25

The Sharp EL-531W calculator may be used on this test.

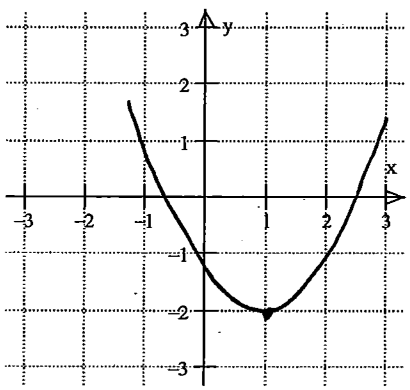
You may not use L'Hôpital's Rule when evaluating limits.

Show all of your work in the space provided.

The number of marks for each question is indicated in brackets.

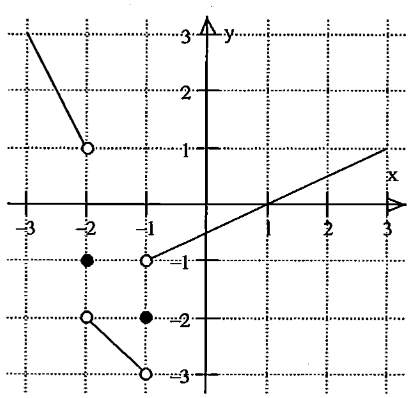
1. Sketch the graph of a **continuous** function $f(x)$ for which $f(1)=-2$ and $f'(1)=0$.
[Note: There are many correct answers.]

[1]



2. Given the graph of the function $f(x)$ below, find $\lim_{x\to -2^+} f(x)$.

[1]



$$\lim_{x\to -2^+} f(x) = -2$$

3. If $f(x)=\begin{cases} 4\cos 7x, & x<\pi/6 \\ 3\sin 4x, & x\geq \pi/6 \end{cases}$, then find $\lim_{x\to(\pi/6)^-} f(x)$. Give the exact value (no decimals).

[1]

$$\lim_{x\to \frac{\pi}{6}^-} f(x) = 4\cos\left(7\cdot\frac{\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

4. Evaluate the limits. If they do not exist, then determine whether they are ∞ or $-\infty$ or neither.

(a) $\lim_{x\to 0}\left[\dfrac{x^4-4x^3+5x^2}{x^2}\right]$ $= \lim_{x\to 0}(x^2-4x+5) = 5$

[1]

(b) $\lim_{x\to 3}\dfrac{\frac{4}{x+1}-1}{x-3}$ $= \lim_{x\to 3}\dfrac{\frac{4-(x+1)}{x+1}}{x-3} = \lim_{x\to 3}\dfrac{3-x}{(x+1)(x-3)} = \lim_{x\to 3}\dfrac{-1}{x+1} = -\dfrac{1}{4}$

[2]

(c) $\lim_{\theta\to 0}\theta\csc 4\theta\cos 4\theta$ $= \lim_{\theta\to 0}\dfrac{\theta}{\sin 4\theta}\cdot\cos 4\theta$

[2]

$= \lim_{\theta\to 0}\dfrac{4\theta}{\sin 4\theta}\cdot\dfrac{\cos 4\theta}{4}$

$= \lim_{\theta\to 0}\dfrac{4\theta}{\sin 4\theta}\cdot\lim_{\theta\to 0}\dfrac{\cos 4\theta}{4}$

$= 1\cdot\dfrac{1}{4} = \dfrac{1}{4}$

(d) $\lim_{x\to -2^-}\dfrac{x^2}{(2x+4)^3}$ $= \dfrac{4}{0^-} = -\infty$

[1]

5. Find all vertical asymptotes and removable discontinuities of the function $f(x)=\dfrac{x^2-10x+25}{x^2-25}$.

[2]

$$f(x) = \frac{(x-5)^2}{(x-5)(x+5)} = \frac{x-5}{x+5}, \quad x \neq \pm 5$$

$x=-5$ is a V.A.

$x=5$ is a removable discontinuity

6.

(a) State the limit definition of the derivative of $f(x)$.

[1] $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

(b) Use the **limit definition** of the derivative to find $f'(x)$, where $f(x)=3\sqrt{2x-5}-9$.

[3]

$$f'(x) = \lim_{\Delta x \to 0} \frac{\left[3\sqrt{2(x+\Delta x)-5}-9\right]-\left[3\sqrt{2x-5}-9\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\sqrt{2x+2\Delta x-5}-3\sqrt{2x-5}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(\sqrt{2x+2\Delta x-5}-\sqrt{2x-5}\right)}{\Delta x} \cdot \frac{\sqrt{2x+2\Delta x-5}+\sqrt{2x-5}}{\sqrt{2x+2\Delta x-5}+\sqrt{2x-5}}$$

$$= \lim_{\Delta x \to 0} \frac{3\left[(2x+2\Delta x-5)-(2x-5)\right]}{\Delta x\left[\sqrt{2x+2\Delta x-5}+\sqrt{2x-5}\right]}$$

$$= \lim_{\Delta x \to 0} \frac{3(2\Delta x)}{\Delta x\left[\sqrt{2x+2\Delta x-5}+\sqrt{2x-5}\right]}$$

$$= \lim_{\Delta x \to 0} \frac{6}{\sqrt{2x+2\Delta x-5}+\sqrt{2x-5}}$$

$$= \frac{6}{\sqrt{2x-5}+\sqrt{2x-5}}$$

$$= \frac{6}{2\sqrt{2x-5}}$$

$$= \frac{3}{\sqrt{2x-5}}$$

7. Let $y = \tan^2 x$. Find $\frac{d^2y}{dx^2}$ by using differentiation rules.

$$\frac{dy}{dx} = 2\tan x \cdot \sec^2 x$$

[2]

$$\frac{d^2y}{dx^2} = 2\tan x\,(2\sec x \cdot \sec x \tan x) + 2\sec^2 x\,(\sec^2 x)$$

$$= 4\sec^2 x \tan^2 x + 2\sec^4 x$$

$$\text{OR} = 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x$$

$$= 6\sec^4 x - 4\sec^2 x$$

8. Find the equation of the tangent line to the curve $y = 1 + 6\sqrt[3]{x^2 + 2x}$ at $x = 2$. Express your answer in the slope-intercept form $y = mx + b$.

$$y = 1 + 6(x^2+2x)^{1/3}$$

$$y' = 2(x^2+2x)^{-2/3}(2x+2)$$

[3]

At $x = 2$, $y = 13$ and $y' = 3$

$\therefore$ point is $(2,13)$ and slope is $m = 3$

$$y - 13 = 3(x-2)$$

$$y - 13 = 3x - 6$$

$$y = 3x + 7$$

9. The position of an object in meters after t seconds is given by $s(t) = 5t^2 - 3t + 19$. After how many minutes is the velocity equal to 27 m/s?

$$V(t) = S'(t) = 10t - 3 = 27$$

$$10t = 30$$

[2]

$$t = 3$$

Velocity is 27 m/s after 3 seconds. or 0.05 minutes.

10. Use the derivative rules to calculate the derivative of $h(t) = \dfrac{(4t^3+5)^3}{(t^2-2)^4}$. Factor the numerator and simplify your answer.

$$h'(t) = \frac{(t^2-2)^4(3)(4t^3+5)^2(12t^2) - (4t^3+5)^3(4)(t^2-2)^3(2t)}{(t^2-2)^8}$$

[3]

$$= \frac{4t(t^2-2)^3(4t^3+5)^2\left[9t(t^2-2) - 2(4t^3+5)\right]}{(t^2-2)^8}$$

$$= \frac{4t(4t^3+5)^2(9t^3-18t-8t^3-10)}{(t^2-2)^5}$$

$$= \frac{4t(4t^3+5)^2(t^3-18t-10)}{(t^2-2)^5}$$