Mathematics 100 Test #1A Name: SOLUTIONS

Instructor: George Ballinger **Section:** ____________

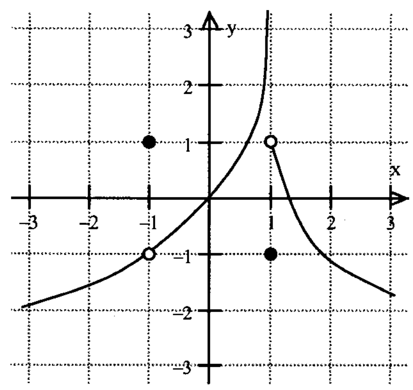
Term: Fall, 2011

Mark: $\overline{25}$

The Sharp EL-531 calculator may be used on this test.
You may not use L'Hôpital's Rule when evaluating limits.
Show all of your work in the space provided.
The number of marks for each question is indicated in brackets.

1. Given the graph of the function $f(x)$ below, find $\lim_{x\to 1^+} f(x)$, or if it does not exist, then determine whether it is ∞ or $-\infty$ or neither.

[1]



$\lim_{x\to 1^+} f(x) = \underline{1}$

2. If $f(x) = \dfrac{1}{1-x^2}$ and $g(x) = \sin x$, then find and simplify the composite function $f \circ g$.

[2]

$(f\circ g)(x) = f(g(x)) = f(\sin x) = \dfrac{1}{1-\sin^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

3. Let $f(x) = \dfrac{1-\cos x}{x}$.

(a) Find the x-values (if any) at which f is not continuous and determine which of the discontinuities is removable.

[2]

f is discontinuous at $x=0$ ($f(0)$ is undefined)

This discontinuity is removable since $\lim_{x\to 0} \dfrac{1-\cos x}{x}$ exists (the limit is 0; it's one of the "special" trig. limits).

(b) Note that $f(-\pi) = -2/\pi < 0$ and $f(\pi) = 2/\pi > 0$. What, if anything, can the Intermediate Value Theorem be used to conclude about the zeros of the function f on the interval $(-\pi, \pi)$? Briefly explain.

[1]

If f were continuous on $[-\pi, \pi]$, then the IVT would let us conclude that $f(c) = 0$ for some $c \in (-\pi, \pi)$.
However, f is not continuous at $x=0$ in the interval (by part (a)) which means the IVT cannot be applied.
So we can't use the IVT to conclude anything about the zeros of f on $(-\pi, \pi)$.

4. Evaluate the following limits and simplify your answer. If the limit does not exist, then determine whether it is ∞ or $-\infty$ or neither. Be sure to show all your work.

(a) $\lim_{x\to 2}\frac{3x^2-7x+2}{x^2-4} = \lim_{x\to 2}\frac{(3x-1)(x-2)}{(x+2)(x-2)} = \lim_{x\to 2}\frac{3x-1}{x+2} = \frac{5}{4}$

[2]

(b) $\lim_{\theta\to 0}\theta\sec\theta\cot\theta = \lim_{\theta\to 0}\theta\cdot\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1$

[2]

(c) $\lim_{x\to 6}\frac{\sqrt{x+3}-3}{x-6} = \lim_{x\to 6}\frac{\sqrt{x+3}-3}{x-6}\cdot\frac{\sqrt{x+3}+3}{\sqrt{x+3}+3} = \lim_{x\to 6}\frac{x+3-9}{(x-6)(\sqrt{x+3}+3)}$

$= \lim_{x\to 6}\frac{x-6}{(x-6)(\sqrt{x+3}+3)} = \lim_{x\to 6}\frac{1}{\sqrt{x+3}+3} = \frac{1}{3+3} = \frac{1}{6}$

[2]

5. Use the **limit definition** of the derivative to find $f'(x)$, where $f(x)=\frac{2}{2-5x}$.

$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{2}{2-5(x+\Delta x)}-\frac{2}{2-5x}}{\Delta x}$

[3]

$= \lim_{\Delta x\to 0}\frac{2(2-5x)-2[2-5(x+\Delta x)]}{[2-5(x+\Delta x)](2-5x)\Delta x} = \lim_{\Delta x\to 0}\frac{4-10x-4+10x+10\Delta x}{[2-5(x+\Delta x)](2-5x)\Delta x}$

$= \lim_{\Delta x\to 0}\frac{10\Delta x}{[2-5(x+\Delta x)](2-5x)\Delta x} = \lim_{\Delta x\to 0}\frac{10}{[2-5(x+\Delta x)](2-5x)}$

$= \frac{10}{(2-5x)(2-5x)} = \frac{10}{(2-5x)^2}$

6. Use derivative rules to calculate the derivatives of the following functions. Simplify your answers.

(a) $f(x) = 4x^3 + 7x^2 - 6x + 3$

$$f'(x) = 12x^2 + 14x - 6$$

[1]

(b) $y = 3\tan^4(5x+2)$

$$y' = 12\tan^3(5x+2) \cdot \sec^2(5x+2) \cdot 5$$
$$= 60\tan^3(5x+2)\sec^2(5x+2)$$

[2]

(c) $y = \dfrac{-2x+1}{4x^2+3}$

$$y' = \frac{(4x^2+3)(-2) - (-2x+1)(8x)}{(4x^2+3)^2} = \frac{-8x^2-6+16x^2-8x}{(4x^2+3)^2}$$

[2]

$$= \frac{8x^2-8x-6}{(4x^2+3)^2} \quad \text{or} \quad \frac{2(2x+1)(2x-3)}{(4x^2+3)^2}$$

7. Find the slope of the tangent line to the curve $y = 6\sqrt{x} - \dfrac{8}{x}$ at $x = 4$.

$$y = 6x^{1/2} - 8x^{-1}$$
$$y' = 3x^{-1/2} + 8x^{-2} = \frac{3}{\sqrt{x}} + \frac{8}{x^2}$$

[2]

$$\text{slope} = \frac{3}{\sqrt{4}} + \frac{8}{4^2} = \frac{3}{2} + \frac{1}{2} = 2$$

8. State the product rule of differentiation and then prove it.

$\frac{d}{dx}[f(x)g(x)] = \underline{f(x)g'(x) + g(x)f'(x)}$

$$\frac{d}{dx}\left[f(x)g(x)\right] = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

[3]

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[f(x+\Delta x) \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x} + g(x) \cdot \frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} f(x+\Delta x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} g(x) \cdot \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= f(x)g'(x) + g(x)f'(x) \quad \blacksquare$$